



chapter 2

Time Value of Money

In 2006, the U.S. pension system was significantly underfunded—companies in the S&P 500 needed an additional \$40 billion to cover their pension commitments. This same situation existed in the U.K. and many other nations. Several factors contributed to this problem, including questionable applications of the time value of money.

First, note that there are two types of pension plans—“defined contribution plans,” where a company provides a specific amount of money, often based on profits, to help its employees when they retire, and “defined benefit plans,” where the company promises to make specific lifetime pension payments to employees when they retire. The payments depend on each employee’s final pay at retirement, years of service with the company, and how long the employees live. Actuaries can estimate these factors, but the company’s future obligations still are uncertain under a defined benefit plan.

As we noted in Chapter 1, a dollar in the future is worth less than a dollar today. Using the time value of money tools in this chapter, we can estimate how much is needed today to make the promised future payments. The amount needed today is actually the present value of the future payments. This present value is called the “pension liability,” because it represents a

claim against the company. Under our pension laws, the company is supposed to set aside enough money each year to meet these future claims. The amount of money that has been set aside, which is invested in a portfolio of stocks, bonds, real estate, and other assets, is called the “pension asset.” If the pension asset is less than the pension liability, then a defined benefit plan is said to be underfunded.

When a company with an underfunded pension plan goes bankrupt, the pension obligations are assumed by the Pension Benefit Guarantee Corporation, a government agency that is implicitly backed by the U.S. government, which means taxpayers. When many companies are underfunded, a very real possibility exists that you, as a taxpayer, will have to bail out the system. Note too that limits exist on the payments to each employee, so if you are highly paid and your employer goes bankrupt, you will also be a loser.

How does the time value of money fit into all this? To see the connection, suppose a plan’s pension assets are initially exactly equal to its liabilities, and then interest rates decline. As you will see, falling interest rates cause the present value of liabilities to increase. Simultaneously, the future earning power of the pension assets will probably be reduced, so unless the company increases its annual contributions the plan will become underfunded. This is exactly what happened

during the early 2000s. Interest rates fell, pension plans became underfunded, companies that were having operating problems were unable to meet their funding requirements, and

as a result many companies failed. This is but one of the many uses of time value analysis, but keep the pension fund example in mind as you go through the chapter.

In Chapter 1, we saw that the primary objective of financial management is to maximize the value of the firm's stock. We also saw that stock values depend in part on the timing of the cash flows investors expect from an investment—a dollar expected soon is worth more than a dollar expected in the distant future. Therefore, it is essential for financial managers to have a clear understanding of the time value of money and its impact on stock prices. These concepts are discussed in this chapter, where we show how the timing of cash flows affects asset values and rates of return.

The principles of time value analysis have many applications, ranging from setting up schedules for paying off loans to decisions about whether to acquire new equipment. *In fact, of all the concepts used in finance, none is more important than the time value of money, also called discounted cash flow (DCF) analysis.* Since time value concepts are used throughout the remainder of the book, it is vital that you understand the material in Chapter 2 and are able to work the chapter problems before you move on to other topics.¹

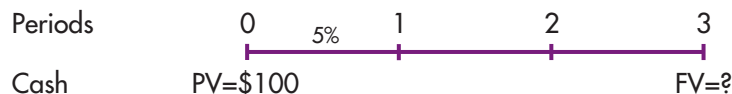
2.1 Time Lines



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The textbook's Web site contains an *Excel* file that will guide you through the chapter's calculations. The file for this chapter is **FM12 Ch 02 Tool Kit.xls**, and we encourage you to open the file and follow along as you read the chapter.

The first step in time value analysis is to set up a **time line**, which will help you visualize what's happening in a particular problem. To illustrate, consider the following diagram, where PV represents \$100 that is on hand today and FV is the value that will be in the account on a future date:



The intervals from 0 to 1, 1 to 2, and 2 to 3 are time periods such as years or months. Time 0 is today, and it is the beginning of Period 1; Time 1 is one period from today, and it is both the end of Period 1 and the beginning of Period 2; and so on. Although the periods are often years, periods can also be quarters or months or even days. Note that each tick mark corresponds to both the *end* of one period and the *beginning* of the next one. Thus, if the periods are years, the tick mark at Time 2 represents both the *end* of Year 2 and the *beginning* of Year 3.

Cash flows are shown directly below the tick marks, and the relevant interest rate is shown just above the time line. Unknown cash flows, which you are trying

¹Calculator manuals tend to be long and complicated, partly because they cover a number of topics that aren't required in the basic finance course. Therefore, we provide, on the textbook's Web site, tutorials for the most commonly used calculators. The tutorials are keyed to this chapter, and they show exactly how to do the required calculations. If you don't know how to use your calculator, go to the Web site, get the relevant tutorial, and go through it as you study the chapter.

to find, are indicated by question marks. Here the interest rate is 5%; a single cash outflow, \$100, is invested at Time 0; and the Time 3 value is an unknown inflow. In this example, cash flows occur only at Times 0 and 3, with no flows at Times 1 or 2. Note that in our example the interest rate is constant for all three years. That condition is generally true, but if it were not then we would show different interest rates for the different periods.

Time lines are essential when you are first learning time value concepts, but even experts use them to analyze complex finance problems, and we use them throughout the book. We begin each problem by setting up a time line to show what's happening, after which we provide an equation that must be solved to find the answer, and then we explain how to use a regular calculator, a financial calculator, and a spreadsheet to find the answer.

SELF-TEST

Do time lines deal only with years or could other periods be used?

Set up a time line to illustrate the following situation: You currently have \$2,000 in a 3-year certificate of deposit (CD) that pays a guaranteed 4% annually.

2.2 Future Values

A dollar in hand today is worth more than a dollar to be received in the future because, if you had it now, you could invest it, earn interest, and end up with more than a dollar in the future. The process of going to **future values (FVs)** from **present values (PVs)** is called **compounding**. To illustrate, refer back to our 3-year time line and assume that you plan to deposit \$100 in a bank that pays a guaranteed 5% interest each year. How much would you have at the end of Year 3? We first define some terms, after which we set up a time line and show how the future value is calculated.

PV = Present value, or beginning amount. In our example, PV = \$100.

FV_N = Future value, or ending amount, of your account after N periods. Whereas PV is the value now, or the *present value*, FV_N is the value N periods into the *future*, after the interest earned has been added to the account.

CF_t = Cash flow. Cash flows can be positive or negative. The cash flow for a particular period is often given a subscript, CF_t , where t is the period. Thus, $CF_0 = PV$ = the cash flow at Time 0, whereas CF_3 would be the cash flow at the end of Period 3.

I = Interest rate earned per year. Sometimes a lowercase i is used. Interest earned is based on the balance at the beginning of each year, and we assume that it is paid at the end of the year. Here $I = 5\%$, or, expressed as a decimal, 0.05. Throughout this chapter, we designate the interest rate as I because that symbol (or I/YR, for interest rate per year) is used on most financial calculators. Note, though, that in later chapters we use the symbol "r" to denote rates because r (for rate of return) is used more often in the finance literature. Note too that in this chapter we generally assume that interest payments are guaranteed by the U.S. government; hence they are certain. In later chapters we will consider risky investments, where the interest rate actually earned might differ from its expected level.

INT = Dollars of interest earned during the year = beginning amount times I. In our example, $INT = \$100(0.05) = \5 .

N = Number of periods involved in the analysis. In our example, $N = 3$. Sometimes the number of periods is designated with a lowercase n, so both N and n indicate number of periods.

Corporate Valuation and the Time Value of Money

In Chapter 1, we told you that managers should strive to make their firms more valuable and that the value of a firm is determined by the size, timing, and risk of its free cash flows (FCF). Recall that free cash flows are the cash flows available for distribution to all of a firm's investors (stockholders and creditors) and that the weighted average cost of capital is the average

rate of return required by all of the firm's investors. We showed you a formula, the same as the one below, for calculating value. That formula takes future cash flows and adjusts them to show how much those future risky cash flows are worth today. That formula is based on time value of money concepts, which we explain in this chapter.

$$\text{Value} = \frac{\text{FCF}_1}{(1 + \text{WACC})^1} + \frac{\text{FCF}_2}{(1 + \text{WACC})^2} + \frac{\text{FCF}_3}{(1 + \text{WACC})^3} + \dots + \frac{\text{FCF}_\infty}{(1 + \text{WACC})^\infty}$$

We can use four different procedures to solve time value problems.² These methods are described in the following sections.

Step-by-Step Approach

The time line used to find the FV of \$100 compounded for 3 years at 5%, along with some calculations, is shown below:

Time	0	1	2	3
Amount at beginning of period	\$100.00	\$105.00	\$110.25	\$115.76

You start with \$100 in the account—this is shown at $t = 0$. Then multiply the initial amount, and each succeeding amount, by $(1 + I) = (1.05)$.

- You earn $\$100(0.05) = \5 of interest during the first year, so the amount at the end of Year 1 (or $t = 1$) is

$$\begin{aligned} \text{FV}_1 &= \text{PV} + \text{INT} \\ &= \text{PV} + \text{PV}(I) \\ &= \text{PV}(1 + I) \\ &= \$100(1 + 0.05) = \$100(1.05) = \$105. \end{aligned}$$

- You begin the second year with \$105, earn $0.05(\$105) = \5.25 on the now larger beginning-of-period amount, and end the year with \$110.25. Interest during Year 2 is \$5.25, and it is higher than the first year's interest, \$5, because you earned $\$5(0.05) = \0.25 interest on the first year's interest. This is called "compounding," and interest earned on interest is called "compound interest."
- This process continues, and because the beginning balance is higher in each successive year, the interest earned each year increases.
- The total interest earned, \$15.76, is reflected in the final balance, \$115.76.

²A fifth procedure is called the tabular approach. It used tables showing "interest factors" and was used before financial calculators and computers became available. Now, though, calculators and spreadsheets such as *Excel* are programmed to calculate the specific factor needed for a given problem and then to use it to find the FV. This is much more efficient than using the tables. Moreover, calculators and spreadsheets can handle fractional periods and fractional interest rates. For these reasons, tables are not used in business today; hence we do not discuss them in the text. For an explanation of the tabular approach, see **Web Extension 2C** at the textbook's Web site.

The step-by-step approach is useful because it shows exactly what is happening. However, this approach is time-consuming, especially if a number of years are involved, so streamlined procedures have been developed.

Formula Approach

In the step-by-step approach above, we multiply the amount at the beginning of each period by $(1 + I) = (1.05)$. Notice that the value at the end of Year 2 is

$$\begin{aligned} FV_2 &= FV_1(1 + I) \\ &= PV(1 + I)(1 + I) \\ &= PV(1 + I)^2 \\ &= 100(1.05)^2 = \$110.25. \end{aligned}$$

If $N = 3$, then we multiply PV by $(1 + I)$ three different times, which is the same as multiplying the beginning amount by $(1 + I)^3$. This concept can be extended, and the result is this key equation:

$$FV_N = PV(1 + I)^N. \quad (2-1)$$

We can apply Equation 2-1 via the formula approach to find the FV in our example:

$$FV_3 = \$100(1.05)^3 = \$115.76.$$

Equation 2-1 can be used with any calculator that has an exponential function, making it easy to find FVs, no matter how many years are involved.

Financial Calculators

Financial calculators are extremely helpful when working time value problems. First, note that financial calculators have five keys that correspond to the five variables in the basic time value equations. We show the inputs for our example above the keys and the output, the FV, below its key. Since there are no periodic payments, we enter 0 for PMT. We describe the keys in more detail below the diagram.

Inputs:	3	5	-100	0	
	N	I/YR	PV	PMT	FV
Output:					115.76

N = Number of periods. Some calculators use n rather than N.

I/YR = Interest rate per period. Some calculators use i or I rather than I/YR.

PV = Present value. In our example we begin by making a deposit, which is an outflow, so the PV should be entered with a negative sign. On most calculators you must enter the 100, then press the $+/-$ key to switch from $+100$ to -100 . If you enter -100 directly, this will subtract 100 from the last number in the calculator and give you an incorrect answer.

PMT = Payment. This key is used if we have a series of equal, or constant, payments. Since there are no such payments in our illustrative problem, we enter $PMT = 0$. We will use the PMT key when we discuss annuities later in this chapter.

FV = Future value. In this example, the FV is positive because we entered the PV as a negative number. If we had entered the 100 as a positive number, then the FV would have been negative.

As noted in our example, you first enter the known values (N, I/YR, PMT, and PV) and then press the FV key to get the answer, 115.76.

Here are some tips for setting up financial calculators. Refer to your calculator manual or to our calculator tutorial on the ThomsonNOW Web site for details on how to set up your specific calculator.

One Payment per Period. Many calculators “come out of the box” assuming that 12 payments are made per year, i.e., they assume monthly payments. However, in this book we generally deal with problems where only 1 payment is made each year. *Therefore, you should set your calculator at 1 payment per year and leave it there.*

End Mode. With most contracts, payments are made at the end of each period. However, some contracts call for payments at the beginning of each period. You can switch between “End Mode” and “Begin Mode,” depending on the problem you are solving. *Since most of the problems in this book call for end-of-period payments, you should return your calculator to End Mode after you work a problem where payments are made at the beginning of periods.*

Number of Decimal Places to Display. Most calculators use all significant digits in all calculations, but display only the number of decimal places that you specify. When working with dollars, we generally specify two decimal places. When dealing with interest rates, we generally specify two places if the rate is expressed as a percentage, like 5.25%, but we specify four places if the rate is expressed as a decimal, like 0.0525.

When students are first learning how to use financial calculators, two mistakes are common. First, students often forget that one cash flow must be negative. Mathematically, financial calculators solve a version of this equation:

$$PV(1 + I)^N + FV_N = 0. \quad (2-2)$$

Notice that for reasonable values of I, either PV or FV_N must be negative and the other must be positive to make the equation true. Intuitively, this is because in all realistic situations, one cash flow is an outflow (which should have a negative sign) and one is an inflow (which should have a positive sign). The bottom line is that one of your inputs for a cash flow must be negative and one must be positive.

The second common mistake often occurs when inputting interest rates. For arithmetic operations with a nonfinancial calculator, the value “5.25%” would be entered as “0.0525.” But with a financial calculator, the value “5.25%” should be entered as “5.25.”

Spreadsheets

Spreadsheet programs are ideally suited for solving many financial problems, including time value of money problems.³ With very little effort, the spreadsheet itself becomes a time line. Figure 2-1 shows how our problem would look in a spreadsheet.

³The textbook’s Web site file *FM12 Ch 02 Tool Kit.xls* does the various calculations using *Excel*. We highly recommend that you go through the models. This will give you practice with *Excel*, which will help tremendously in later courses, in the job market, and in the workplace. Also, going through the models will enhance your understanding of financial concepts.

Figure 2-1

Summary: Future Value Calculations

	A	B	C	D	E	F	G
12							
13							
14	Investment	= CF ₀ = PV =	-\$100.00				
15	Interest rate	= I =	5.00%				
16	No. of periods	= N =	3				
17			Periods:	0	1	2	3
18			Cash Flow Time Line:	-\$100	→	→	→ FV = ?
19							
20							
21	<u>Step-by-Step Approach:</u>			\$100	→	\$105.00	→ \$110.25
22							→ \$115.76
23	<u>Formula Approach:</u>	$FV_N = PV(1+I)^N$		$FV_N =$	$\$100(1.05)^3$	$=$	\$115.76
24							
25				3	5	-\$100.00	\$0
26	<u>Calculator Approach:</u>			N	I/YR	PV	FV
27							\$115.76
28							
29	<u>Excel Approach:</u>		FV Function:	$FV_N =$	$=FV(I,N,0,PV)$		
30			Fixed inputs:	$FV_N =$	$=FV(0.05,3,0,-100)$		\$115.76
31			Cell references:	$FV_N =$	$=FV(C15,C16,0,C14)$		\$115.76
32	In the Excel formula, the terms are entered in this sequence: interest, periods, 0 to indicate no intermediate cash flows, and then the PV. The data can be entered as fixed numbers or as cell references.						

We used *Excel* to create Figure 2-1, which summarizes the four methods for finding the FV. Note that spreadsheets can be used to do calculations, but they can also be used like a word processor to create exhibits like Figure 2-1, which includes text, drawings, and calculations. The letters across the top designate columns, the numbers to the left designate rows, and the rows and columns jointly designate cells.

It is a good practice to put a problem’s inputs in the same section. For example, in Figure 2-1, C14 is the cell where we specify the -\$100 investment, C15 shows the interest rate, and C16 shows the number of periods.

Drawing a time line is an important step in solving finance problems. When we work a problem by hand we usually draw a time line. When we work a problem in *Excel*, we usually put in a time line. For example, in Figure 2-1 the time line is in Rows 17 to 19. Notice how easy it is in *Excel* to put in a time line, with each column designating a different period on the time line.

In Row 21 we have *Excel* go through the step-by-step calculations, multiplying the beginning-of-year values by (1 + I) to find the compounded value at the end of each period. Cell G21 shows the final result of the step-by-step approach.

We illustrate the formula approach in Row 23, using *Excel* to solve Equation 2-1 and find the FV. Cell G23 shows the formula result, \$115.76.

Rows 25 to 27 illustrate the inputs and result from using a financial calculator.

The last section of Figure 2-1 illustrates *Excel*’s FV function. You can access the function wizard by clicking the f_x symbol in *Excel*’s formula bar, or you can go to the menu bar, select Insert, and then select Function from the drop-down menu. Select the category for Financial functions, and then select the FV function. The function is $=FV(I,N,0,PV)$, as shown in Cell E29.⁴ Cell E30 shows how the formula would look



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

⁴The third entry in the FV function is zero in this example, to indicate that there are no periodic payments. Later in this chapter we will use the function in situations where we do have periodic payments.

The Power of Compound Interest



Suppose you are 26 years old and just received your MBA. After reading the introduction to this chapter, you decide to start investing in the stock market for your retirement. Your goal is to have \$1 million when you retire at age 65. Assuming you earn a 10% annual rate on your stock investments, how much must you invest at the end of each year in order to reach your goal?

The answer is \$2,490.98, but this amount depends critically on the return earned on your investments. If returns drop to 8%, your required annual contributions will rise to \$4,185.13. If returns rise to 12%, you will need to put away only \$1,461.97 per year.

What if you are like most of us and wait until later to worry about retirement? If you wait until age

40, you will need to save \$10,168 per year to reach your \$1 million goal, assuming you earn 10%, and \$13,679 per year if you earn only 8%. If you wait until age 50 and then earn 8%, the required amount will be \$36,830 per year.

While \$1 million may seem like a lot of money, it won't be when you get ready to retire. If inflation averages 5% a year over the next 39 years, your \$1 million nest egg will be worth only \$149,148 in today's dollars. At an 8% rate of return, and assuming you live for 20 years after retirement, your annual retirement income in today's dollars will be only \$15,191 before taxes. So, after celebrating graduation and your new job, start saving!

with numbers as inputs; the actual function itself is in Cell G30. Cell E31 shows how the formula would look with cell references as inputs, with the actual function in Cell G31. We always recommend using cell references as inputs to functions, because this makes it easy to change inputs and see the effects on the output.

Notice that when entering interest rates in *Excel*, you must input the actual number. For example, in cell C15, we input "0.05," and then formatted it as a percentage. In the function itself, you can enter "0.05" or "5%," but if you enter "5," *Excel* will think you mean 500%. This is exactly opposite the convention for financial calculators.

Comparing the Procedures

The first step in solving any time value problem is to understand the verbal description of the problem well enough to diagram it on a time line. Woody Allen said that 90% of success is just showing up. With time value problems, 90% of success is correctly setting up the time line.

After you diagram the problem on a time line, your next step is to pick an approach to solve the problem. Which of the approaches should you use? The answer depends on the particular situation.

All business students should know Equation 2-1 by heart and should also know how to use a financial calculator. So, for simple problems such as finding the future value of a single payment, it is probably easiest and quickest to use either the formula approach or a financial calculator.

For problems with more than a couple of cash flows, the formula approach is usually too time-consuming, so here either the calculator or spreadsheet approaches would generally be used. Calculators are portable and quick to set up, but if many calculations of the same type must be done, or if you want to see how changes in an input such as the interest rate affect the future value, the spreadsheet approach is generally more efficient. If the problem has many irregular cash flows, or if you want to analyze many scenarios with different cash flows, then the spreadsheet approach is definitely the most efficient.



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

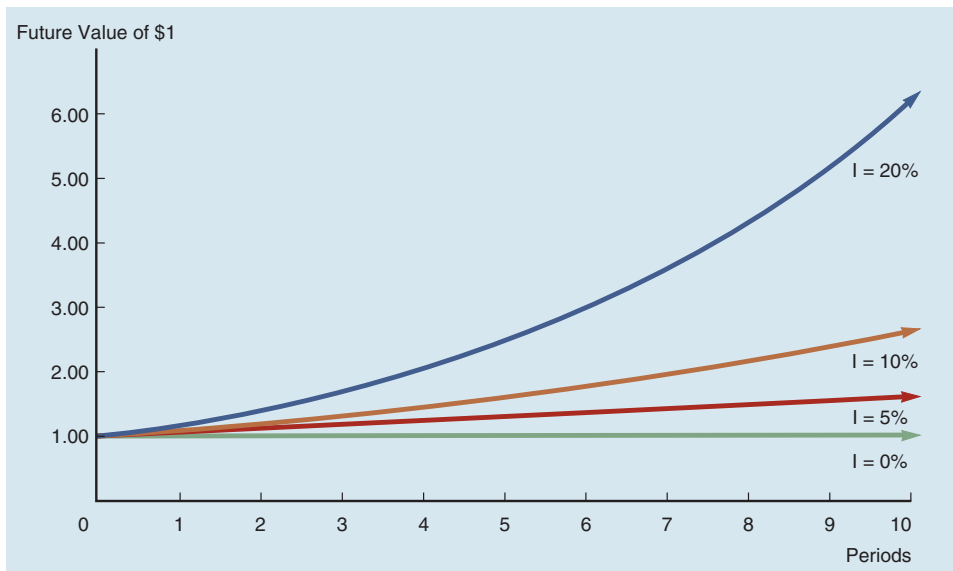
The important thing is that you understand the various approaches well enough to make a rational choice, given the nature of the problem and the equipment you have available. In any event, you must understand the concepts behind the calculations and know how to set up time lines in order to work complex problems. This is true for stock and bond valuation, capital budgeting, lease analysis, and many other important financial problems.

Graphic View of the Compounding Process

Figure 2-2 shows how a \$1 investment grows over time at different interest rates. We made the curves by solving Equation 2-1 with different values for N and I. The interest rate is a growth rate: If a sum is deposited and earns 5% interest per year, then the funds on deposit will grow by 5% per year. Note also that time value concepts can be applied to anything that grows—sales, population, earnings per share, or your future salary.

Figure 2-2

Growth of \$1 at Various Interest Rates and Time Periods



SELF-TEST

Explain why this statement is true: "A dollar in hand today is worth more than a dollar to be received next year."

What is compounding? What would the future value of \$100 be after 5 years at 10% *compound* interest? (\$161.05)

Suppose you currently have \$2,000 and plan to purchase a 3-year certificate of deposit (CD) that pays 4% interest compounded annually. How much will you have when the CD matures? How would your answer change if the interest rate were 5%, or 6%, or 20%? (\$2,249.73; \$2,315.25; \$2,382.03; \$3,456.00) (Hint: With a calculator, enter N = 3, I/YR = 4, PV = -2000, and PMT = 0, then press FV to get 2,249.73. Then, enter I/YR = 5 to override the 4% and press FV again to get the second answer. In general, you can change one input at a time to see how the output changes.)

A company's sales in 2005 were \$100 million. If sales grow at 8%, what will they be 10 years later, in 2015? (\$215.89 million)

How much would \$1, growing at 5% per year, be worth after 100 years? What would FV be if the growth rate were 10%? (\$131.50; \$13,780.61)

2.3 Present Values

Suppose you have some extra money to invest. A broker offers to sell you a security that will pay a guaranteed \$115.76 three years from now. Banks are currently offering a guaranteed 5% interest on 3-year certificates of deposit (CDs), and if you don't buy the security you will buy a CD. The 5% rate paid on the CD is defined as your **opportunity cost**, or the rate of return you could earn on an alternative investment of similar risk. Given these conditions, what's the most you should pay for the security?

First, recall from the future value example in the last section that if you invested \$100 at 5% it would grow to \$115.76 in 3 years. You would also have \$115.76 after 3 years if you bought the broker's security. Therefore, the most you should pay for the security is \$100—this is its “fair price.” This is also equal to the intrinsic, or fundamental, value of the security. If you could buy the security for *less than* \$100, you should buy it rather than invest in the CD. Conversely, if its price were *more than* \$100, you should buy the CD. If the security's price were exactly \$100, you should be indifferent between the security and the CD.

The \$100 is defined as the present value, or PV, of \$115.76 due in 3 years when the appropriate interest rate is 5%. In general, *the present value of a cash flow due N years in the future is the amount which, if it were on hand today, would grow to equal the given future amount.* Since \$100 would grow to \$115.76 in 3 years at a 5% interest rate, \$100 is the present value of \$115.76 due in 3 years at a 5% rate.

Finding present values is called **discounting**, and as noted above, it is the reverse of compounding—if you know the PV, you can compound to find the FV, while if you know the FV, you can discount to find the PV. Indeed, we simply solve Equation 2-1, the formula for the future value, for the PV to produce the basic present value equation, 2-3:

$$\text{Future value} = FV_N = PV(1 + I)^N, \quad (2-1)$$

$$\text{Present value} = PV = \frac{FV_N}{(1 + I)^N}. \quad (2-3)$$

The top section of Figure 2-3 shows the time line and calculates the PV using the step-by-step approach. When we found the future value in the previous section, we worked from left to right, multiplying the initial amount and each subsequent amount by $(1 + I)$. To find present values, we work backwards, or from right to left, dividing the future value and each subsequent amount by $(1 + I)$. This procedure shows exactly what's happening, and that can be quite useful when you are working complex problems. However, it's inefficient, especially if you are dealing with a number of years.

With the formula approach we use Equation 2-3, simply dividing the future value by $(1 + I)^N$. This is more efficient than the step-by-step approach, and it gives the same result, as shown in Figure 2-3, Row 73.

Equation 2-2 is built into financial calculators, and as shown in Figure 2-3, Row 76, we can find the PV by entering values for $N = 3$, $I/YR = 5$, $PMT = 0$, and $FV = 115.76$, and then pressing the PV key to get -100 .



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

⁵The third entry in the PV function is zero to indicate that there are no intermediate payments in this particular example.

Figure 2-3

Summary: Present Value Calculations

	A	B	C	D	E	F	G
62	Figure 2-3. Summary of Present Value Calculations						
63							
64	Future payment = CF _N = FV =		\$115.76				
65	Interest rate = I =		5.00%				
66	No. of periods = N =		3				
67		Periods:	0	1	2	3	
68		Cash Flow Time Line:	PV = ?	←	←	←	\$115.76
69							
70							
71	Step-by-Step Approach:		\$100.00	←	\$105.00	←	\$110.25
72							\$115.76
73	Formula Approach: PV = FV _N / (1 + I) ^N				PV = \$115.76 / (1.05) ³	=	\$100.00
74							
75			3	5		\$0	\$115.76
76	Calculator Approach:		N	I/YR	PV	PMT	FV
77					-\$100.00		
78							
79	Excel Approach:	PV Function:	PV =	=PV(I,N,0,FV)			
80		Fixed inputs:	PV =	=PV(0.05,3,0,115.76)		=	-\$100.00
81		Cell references:	PV =	=PV(C65,C66,0,C64)		=	-\$100.00
82							
83	In the Excel function, 0 indicates that there are no intermediate cash flows.						

Spreadsheets also have a function that solves Equation 2-2. In *Excel*, this is the PV function, and it is written as =PV(I,N,0,FV).⁵ Cell E79 shows the inputs to this function. Cell E80 shows the function with fixed numbers as inputs, with the actual function and the resulting -\$100 in Cell G80. Cell E81 shows the function with cell references as inputs, with the actual function and the resulting -\$100 in Cell G81.

The fundamental goal of financial management is to maximize the firm's value, and the value of a business (or any asset, including stocks and bonds) is the *present value* of its expected future cash flows. Since present value lies at the heart of the valuation process, we will have much more to say about it in the remainder of this chapter and throughout the book.

Graphic View of the Discounting Process

Figure 2-4 shows that the present value of a sum to be received in the future decreases and approaches zero as the payment date is extended further and further into the future, and also that the present value falls faster the higher the interest rate. At relatively high rates, funds due in the future are worth very little today, and even at relatively low rates present values of sums due in the very distant future are quite small. For example, at a 20% discount rate, \$1 million due in 100 years would be worth only about 1 cent today. (However, 1 cent would grow to almost \$1 million in 100 years at 20%.)

SELF-TEST

What is "discounting," and how is it related to compounding? How is the future value equation (2-1) related to the present value equation (2-3)?

How does the present value of a future payment change as the time to receipt is lengthened? As the interest rate increases?

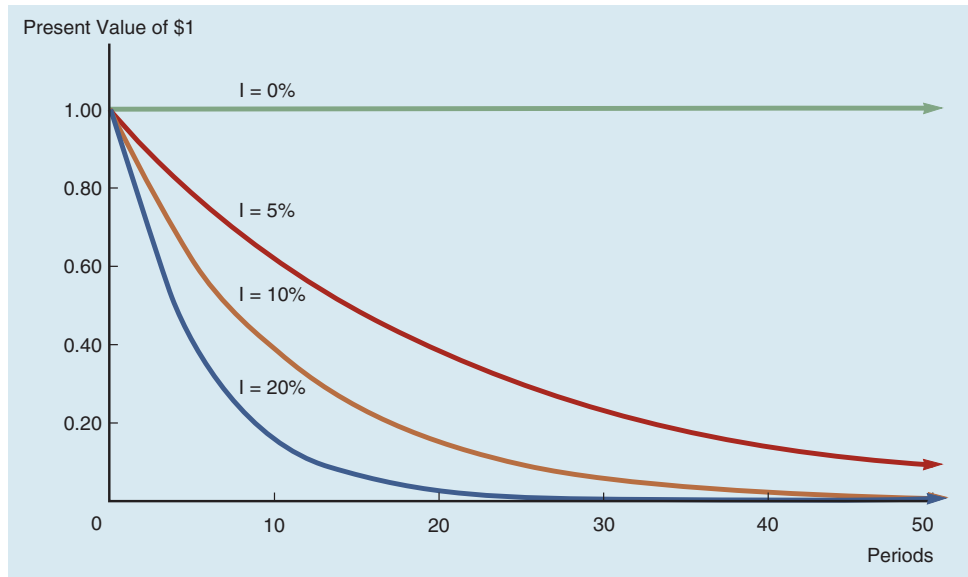
(continued)

SELF-TEST

Suppose a U.S. government bond promises to pay \$2,249.73 three years from now. If the going interest rate on 3-year government bonds is 4%, how much is the bond worth today? How would your answer change if the bond matured in 5 rather than 3 years? What if the interest rate on the 5-year bond were 6% rather than 4%? (\$2,000; \$1,849.11; \$1,681.13)

How much would \$1,000,000 due in 100 years be worth today if the discount rate were 5%? If the discount rate were 20%? (\$7,604.49; \$0.0121)

Figure 2-4 Present Value of \$1 at Various Interest Rates and Time Periods



2.4 Finding the Interest Rate, I

Thus far we have used Equations 2-1, 2-2, and 2-3 to find future and present values. Those equations have four variables, and if we know three of them, we can solve for the fourth. Thus, if we know PV, I, and N, then we can solve Equation 2-2 for FV, while if we know FV, I, and N we can solve Equation 2-2 to find PV. That's what we did in the preceding two sections.

Now suppose we know PV, FV, and N, and we want to find I. For example, suppose we know that a given security has a cost of \$100 and that it will return \$150 after 10 years. Thus, we know PV, FV, and N, and we want to find the rate of return we will earn if we buy the security. Here's the solution using Equation 2-1:

$$\begin{aligned}
 FV &= PV(1 + I)^N \\
 \$150 &= \$100(1 + I)^{10} \\
 \$150/\$100 &= (1 + I)^{10} \\
 (1 + I)^{10} &= 1.5 \\
 (1 + I) &= 1.5^{(1/10)} \\
 (1 + I) &= 1.0414 \\
 I &= 0.0414 = 4.14\%.
 \end{aligned}$$



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

Finding the interest rate by solving the formula takes a little time, but financial calculators and spreadsheets can find interest rates almost instantly. Here's the calculator setup:

Inputs:	10	-100	0	150
	N	I/YR	PV	PMT
Output:	4.14			

Enter $N = 10$, $PV = -100$, $PMT = 0$ because there are no payments until the security matures, and $FV = 150$. Then, when you press the I/YR key, the calculator gives the answer, 4.14%. Notice that the PV is a negative value because it is a cash outflow (an investment) and the FV is positive because it is a cash inflow (a return of the investment). If you enter both PV and FV as positive numbers (or both as negative numbers), your calculator will be unable to solve for the interest rate.

In *Excel*, the RATE function can be used to find the interest rate: $\text{=RATE}(N,PMT,PV,FV)$. For this example, the function would be $\text{=RATE}(10,0,-100,150) = 0.0414 = 4.14\%$. See the file *FM12 Ch 02 Tool Kit.xls* at the textbook's Web site for an example.

SELF-TEST

The U.S. Treasury offers to sell you a bond for \$585.43. No payments will be made until the bond matures 10 years from now, at which time it will be redeemed for \$1,000. What interest rate would you earn if you bought this bond for \$585.43? What rate would you earn if you could buy the bond for \$550? For \$600? (5.5%; 6.16%; 5.24%)

Microsoft earned \$0.12 per share in 1994. Ten years later, in 2004, it earned \$1.04. What was the growth rate in Microsoft's earnings per share (EPS) over the 10-year period? If EPS in 2004 had been \$0.65 rather than \$1.04, what would the growth rate have been? (24.1%; 18.41%)

2.5 Finding the Number of Years, N

We sometimes need to know how long it will take to accumulate a sum of money, given our beginning funds and the rate we will earn on those funds. For example, suppose we now have \$500,000 and the interest rate is 4.5%. How long will it take to grow to \$1 million?

Here is Equation 2-1:

$$\$1,000,000 = \$500,000(1 + 0.045)^N.$$

Transform this to

$$\$1,000,000/\$500,000 = 2 = (1 + 0.045)^N.$$

Take the natural log of both sides, and then solve for N:

$$\begin{aligned} N[\ln(1.045)] &= \ln(2) \\ N &= \ln(2)/\ln(1.045). \end{aligned}$$

Find the logs with a calculator, and complete the solution:

$$\begin{aligned} N &= 0.6931/0.0440 \\ &= 15.7473. \end{aligned}$$



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

Therefore, it takes \$500,000 about 15.7473 years to grow to \$1,000,000 if the interest rate is 4.5%.

As you might expect, financial calculators and spreadsheets can find N very quickly. Here's the calculator setup:

Inputs:	4.5	-500000	0	1000000
	N	I/YR	PV	PMT
Output:	15.7473			

Enter $I/YR = 4.5$, $PV = -500000$, $PMT = 0$, and $FV = 1000000$. When we press the N key, we get the answer, 15.7473 years.

In *Excel*, the `NPER` function can be used to find the number of periods: `=NPER(I,PMT,PV,FV)`. For this example, the function would be `=RATE(0.045,0,-500000,1000000) = 15.7473`. See the file *FM12 Ch 02 Tool Kit.xls* at the textbook's Web site for an example.

SELF-TEST

How long would it take \$1,000 to double if it were invested in a bank that pays 6% per year? How long would it take if the rate were 10%? (11.9 years; 7.27 years)

Microsoft's 2004 earnings per share were \$1.04, and its growth rate during the prior 10 years was 24.1% per year. If that growth rate were maintained, how long would it take for Microsoft's EPS to double? (3.21 years)

2.6 Annuities

Thus far we have dealt with single payments, or "lump sums." However, many assets provide a series of cash inflows over time, and many obligations like auto loans, student loans, and mortgages require a series of payments. If the payments are equal and are made at fixed intervals, then the series is an **annuity**. For example, \$100 paid at the end of each of the next 3 years is a 3-year annuity.

If payments occur at the *end* of each period, then we have an **ordinary** (or **deferred**) **annuity**. Payments on mortgages, car loans, and student loans are examples of ordinary annuities. If the payments are made at the *beginning* of each period, then we have an **annuity due**. Rental payments for an apartment, life insurance premiums, and lottery payoffs (if you are lucky enough to win one!) are examples of annuities due. Ordinary annuities are more common in finance, so when we use the term "annuity" in this book, assume that the payments occur at the ends of the periods unless otherwise noted.

Here are the time lines for a \$100, 3-year, 5%, ordinary annuity and for the same annuity on an annuity due basis. With the annuity due, each payment is shifted back to the left by 1 year. A \$100 deposit will be made each year, so we show the payments with minus signs.

Ordinary Annuity:

Periods	0	1	2	3
		5%		
Payments		-\$100	-\$100	-\$100

Annuity Due:

As we demonstrate in the following sections, we can find an annuity's future and present values, the interest rate built into annuity contracts, and how long it takes to reach a financial goal using an annuity. Keep in mind that annuities must have *constant payments* and a *fixed number of periods*. If these conditions don't hold, then we don't have an annuity.

SELF-TEST

What is the difference between an ordinary annuity and an annuity due?

Why should you rather receive an annuity due for \$10,000 per year for 10 years than an otherwise similar ordinary annuity?

2.7 Future Value of an Ordinary Annuity

Consider the ordinary annuity diagrammed previously, where you deposit \$100 at the end of each year for 3 years and earn 5% per year. Figure 2-5 calculates the **future value of the annuity**, FVA_N , using each of the approaches we used for single cash flows.

As shown in the step-by-step section of Figure 2-5, we compound each payment out to Time 3, then sum those compounded values to find the annuity's FV, $FVA_3 = \$315.25$. The first payment earns interest for two periods, the second for one period, and the third earns no interest at all because it is made at the end of the annuity's life. This approach is straightforward, but if the annuity extends out for many years, it is cumbersome and time-consuming.

As you can see from the time line diagram, with the step-by-step approach we apply the following equation, with $N = 3$ and $I = 5\%$:

$$\begin{aligned} FVA_N &= PMT(1 + I)^{N-1} + PMT(1 + I)^{N-2} + PMT(1 + I)^{N-3} \\ &= \$100(1.05)^2 + \$100(1.05)^1 + \$100(1.05)^0 \\ &= \$315.25. \end{aligned}$$

For the general case, the future value of an annuity is

$$\begin{aligned} FVA_N &= PMT(1 + I)^{N-1} + PMT(1 + I)^{N-2} \\ &\quad + PMT(1 + I)^{N-3} + \dots + PMT(1 + I)^0. \end{aligned}$$

As shown in **Web Extension 2A** at the textbook's Web site, the future value of an annuity can be written as⁶

$$FVA_N = PMT \left[\frac{(1 + I)^N}{I} - \frac{1}{I} \right]. \quad (2-4)$$

⁶Section 2.11 shows that the present value of an infinitely long annuity, called a perpetuity, is equal to PMT/I . The cash flows of an ordinary annuity of N periods are equal to the cash flows of a perpetuity minus the cash flows of a perpetuity that begins at year $N + 1$. Therefore, the future value of an N -period annuity is equal to the future value (as of year N) of a perpetuity minus the value (as of year N) of a perpetuity that begins at year $N + 1$. See **Web Extension 2A** at the textbook's Web site for details of this derivation.

Figure 2-5

Summary: Future Value of an Ordinary Annuity

	A	B	C	D	E	F	G
131	Payment amount	= PMT =	\$100.00				
132	Interest rate	= I =	5.00%				
133	Number of periods	= N =	3				
134							
135		Periods:	0	1	2	3	
136			----- ----- -----				
137		Cash Flow Time Line:		-\$100	-\$100	-\$100	
138							
139		Step-By-Step Approach.					
140		Multiply each payment by					
141		$(1+I)^{N-t}$ and sum these FVs to					
142		find FVA_N :					
143							
144		Formula Approach:					
145							
146		$FVA_N =$	$PMT \times \left(\frac{(1+I)^N}{I} - \frac{1}{I} \right)$	=		\$315.25	
147							
148							
149							
150		Calculator Approach:					
151							
152							
153		Excel Function Approach:					
154							
155							

Using this formula, the future value of the annuity is

$$FVA_3 = \$100 \left[\frac{(1 + 0.05)^3}{0.05} - \frac{1}{0.05} \right] = \$315.25.$$

Annuity problems are much easier to solve with financial calculators and spreadsheets. The following formula is built into financial calculators and spreadsheets:

$$PV(1 + I)^N + PMT \left[\frac{(1 + I)^N}{I} - \frac{1}{I} \right] + FV = 0. \tag{2-5}$$



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

When solving an annuity problem with a financial calculator, the presence of recurring payments means that we use the PMT key. Here’s the calculator setup for our illustrative annuity:

Inputs:	3	5	0	-100	End Mode
	N	I/YR	PV	PMT	FV
Output:					315.25

We enter $PV = 0$ because we start off with nothing, and we enter $PMT = -100$ because we plan to deposit this amount in the account at the end of each year. When we press the FV key we get the answer, $FVA_3 = 315.25$.

Since this is an ordinary annuity, with payments coming at the *end* of each year, we must set the calculator appropriately. As noted earlier, calculators “come out of the box” set to assume that payments occur at the end of each period, that is, to deal with ordinary annuities. However, there is a key that enables us to switch between ordinary annuities and annuities due. For ordinary annuities, the designation is “End Mode” or something similar, while for annuities due the designator is “Begin,” “Begin Mode,” or “Due” or something similar. If you make a mistake and set your calculator on Begin Mode when working with an ordinary annuity, then each payment will earn interest for one extra year, which will cause the compounded amounts, and thus the FVA, to be too large.

The last approach in Figure 2-5 shows the spreadsheet solution, using *Excel's* built-in FV function: $=FV(I,N,PMT,PV)$. In our example, we have $=FV(0.05,3,-100,0)$, with a resulting value of \$315.25.

SELF-TEST

For an ordinary annuity with 5 annual payments of \$100 and a 10% interest rate, how many years will the 1st payment earn interest, and what will this payment's value be at the end? Answer this same question for the 5th payment. (4 years, \$146.41; 0 years, \$100)

Assume that you plan to buy a condo 5 years from now, and you estimate that you can save \$2,500 per year. You plan to deposit the money in a bank that pays 4% interest, and you will make the first deposit at the end of the year. How much will you have after 5 years? How would your answer change if the interest rate were increased to 6%? Lowered to 3%? (\$13,540.81; \$14,092.73; \$13,272.84)

2.8 Future Value of an Annuity Due

Since each payment occurs one period earlier with an annuity due, the payments will all earn interest for one additional period. Therefore, the FV of an annuity due will be greater than that of a similar ordinary annuity.

If you went through the step-by-step procedure, you would see that our illustrative annuity due has an FV of \$331.01 versus \$315.25 for the ordinary annuity. See *FM12 Ch 02 Tool Kit.xls* at the textbook's Web site for a summary of these calculations.

With the formula approach, we first use Equation 2-4, but since each payment occurs one period earlier, we multiply the Equation 2-4 result by $(1 + I)$:

$$FVA_{\text{due}} = FVA_{\text{ordinary}}(1 + I). \quad (2-6)$$

Thus, for the annuity due, $FVA_{\text{due}} = \$315.25(1.05) = \331.01 , which is the same result as found using the step-by-step approach.

With a calculator we input the variables just as we did with the ordinary annuity, but now we set the calculator to Begin Mode to get the answer, \$331.01.

Inputs:	3	5	0	-100	Begin Mode
	N	I/YR	PV	PMT	FV
Output:					331.01



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See *FM12 Ch 02 Tool Kit.xls* for all calculations.

In *Excel*, we still use the FV function, but we must indicate that we have an annuity due. The function is **=FV(I,N,PMT,PV,Type)**, where Type indicates the type of annuity. If Type is omitted, *Excel* assumes that it is zero, which indicates an ordinary annuity. For an annuity due, Type = 1. As shown in *FM12 Ch 02 Tool Kit.xls* at the textbook's Web site, the function is **=FV(0.05,3,-100,0,1) = \$331.01**.

SELF-TEST

Why does an annuity due always have a higher future value than an ordinary annuity?

If you calculated the value of an ordinary annuity, how could you find the value of the corresponding annuity due?

Assume that you plan to buy a condo 5 years from now, and you need to save for a down payment. You plan to save \$2,500 per year, with the first payment made *immediately*, and you will deposit the funds in a bank account that pays 4%. How much will you have after 5 years? How much would you have if you made the deposits at the end of each year? (\$14,082.44; \$13,540.81)

2.9 Present Value of an Ordinary Annuity and of an Annuity Due

The present value of an annuity, PVA_N , can be found using the step-by-step, formula, calculator, or spreadsheet methods. We begin with an ordinary annuity.

Present Value of an Ordinary Annuity

See Figure 2-6 for a summary of the different approaches for calculating the present value of an ordinary annuity.

As shown in the step-by-step section of Figure 2-6, we discount each payment back to Time 0, then sum those discounted values to find the annuity's PV, $PVA_3 = \$272.32$. This approach is straightforward, but if the annuity extends out for many years, it is cumbersome and time-consuming.

The time line diagram shows that with the step-by-step approach we apply the following equation, with $N = 3$ and $I = 5\%$:

$$PVA_N = PMT/(1 + I)^1 + PMT/(1 + I)^2 + \cdots + PMT/(1 + I)^N.$$

As shown in *Web Extension 2A* at the textbook's Web site, the present value of an annuity can be written as⁷

$$PVA_N = PMT \left[\frac{1}{I} - \frac{1}{I(1 + I)^N} \right]. \quad (2-7)$$

For our annuity, this present value is

$$PVA_3 = PMT \left[\frac{1}{0.05} - \frac{1}{0.05(1 + 0.05)^3} \right] = \$272.32.$$

⁷Section 2.11 shows that the present value of an infinitely long annuity, called a perpetuity, is equal to PMT/I . The cash flows of an ordinary annuity of N periods are equal to the cash flows of a perpetuity minus the cash flows of a perpetuity that begins at year $N + 1$. Therefore, the present value of an N -period annuity is equal to the present value of a perpetuity minus the present value of a perpetuity that begins at year $N + 1$. See *Web Extension 2A* at the textbook's Web site for details of this derivation.



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See *FM12 Ch 02 Tool Kit.xls* for all calculations.



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See *FM12 Ch 02 Tool Kit.xls* for all calculations.

Figure 2-6

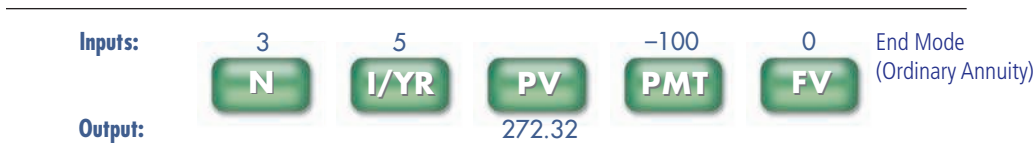
Summary: Present Value of an Ordinary Annuity

	A	B	C	D	E	F	G
196	Payment amount	= PMT =	\$100.00				
197	Interest rate	= I =	5.00%				
198	Number of periods	= N =	3				
199							
200		Periods:	0	1	2	3	
201							
202		Cash Flow Time Line:					
203							
204							
205							
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215							
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217							
218							
219							
220							

204	Step-by-Step Approach.	\$95.24	←
205	Divide each payment by	\$90.70	←
206	(1 + I) ^t and sum these PVs to	\$86.38	←
207	find PVA _N :	<u>\$272.32</u>	

209	Formula Approach:				
210					
211	PVA _N	=	$PMT \times \left(\frac{1}{I} - \frac{1}{I(1+I)^N} \right)$	=	\$272.32
212					
213					
214					
215	Calculator Approach:				
216					
217					
218	Excel Function Approach:				
219					
220					

Financial calculators are programmed to solve Equation 2-5, so we merely input the variables and press the PV key, *making sure the calculator is set to End Mode*. The calculator setup is shown below:



The last approach in Figure 2-6 shows the spreadsheet solution, using *Excel's* built-in PV function: =PV(L,N,PMT,FV). In our example, we have =PV(0.05,3,-100,0), with a resulting value of \$272.32.

Present Value of Annuities Due

Since each payment for an annuity due occurs one period earlier, the payments will all be discounted for one less period. Therefore, the PV of an annuity due will be greater than that of a similar ordinary annuity.

If you went through the step-by-step procedure, you would see that our illustrative annuity due has a PV of \$285.94 versus \$272.32 for the ordinary annuity. See *FM12 Ch 02 Tool Kit.xls* at the textbook's Web site for a summary of these calculations.

With the formula approach, we first use Equation 2-7, but since each payment occurs one period earlier, we multiply the Equation 2-7 result by $(1 + I)$:

$$PVA_{\text{due}} = PVA_{\text{ordinary}}(1 + I). \quad (2-8)$$

Thus, for the annuity due, $PVA_{\text{due}} = \$272.32(1.05) = \285.94 , which is the same result as found using the step-by-step approach.

With a financial calculator, the inputs are the same as for an ordinary annuity, except you must set the calculator to the Begin Mode.

Inputs:	3	5	-100	0	Begin Mode (Annuity Due)
	N	I/YR	PV	PMT	
Output:			285.94		

In *Excel*, we still use the PV function, but we must indicate that we have an annuity due. The function is `=PV(I,N,PMT,FV,Type)`, where Type indicates the type of annuity. If Type is omitted, *Excel* assumes that it is zero, which indicates an ordinary annuity. For an annuity due, Type = 1. As shown in *FM12 Ch 02 Tool Kit.xls* at the ThomsonNOW Web site, the function for this example is `=PV(0.05,3,-100,0,1) = $285.94`.

SELF-TEST

Why does an annuity due have a higher present value than an ordinary annuity?

If you know the present value of an ordinary annuity, how can you find the PV of the corresponding annuity due?

What is the PVA of an ordinary annuity with 10 payments of \$100 if the appropriate interest rate is 10%? What would PVA be if the interest rate were 4%? What if the interest rate were 0%? How would the PVA values differ if we were dealing with annuities due? (\$614.46; \$811.09; \$1,000.00; \$675.90; \$843.53; \$1,000.00)

Assume that you are offered an annuity that pays \$100 at the end of each year for 10 years. You could earn 8% on your money in other investments with equal risk. What is the most you should pay for the annuity? If the payments began immediately, how much would the annuity be worth? (\$671.01; \$724.69)

2.10 Finding Annuity Payments, Periods, and Interest Rates

We can find payments, periods, and interest rates for annuities. Five variables come into play: N, I, PMT, FV, and PV. If we know any four, we can find the fifth. It is not too difficult to solve Equation 2-5 for the payment if you know all of the other inputs. But when trying to find the interest rate or number of periods, Equation 2-5 usually must be solved by trial-and-error. Therefore, we present only the solution approaches using a financial calculator or spreadsheet.

Finding Annuity Payments, PMT

Suppose we need to accumulate \$10,000 and have it available five years from now. Suppose further that we can earn a return of 6% on our savings, which are currently zero. Thus, we know that $FV = 10,000$, $PV = 0$, $N = 5$, and $I/YR = 6$. We can enter these values in a financial calculator and then press the PMT key to find how large our deposits must be. The answer will, of course, depend on whether we make deposits at the end of each year (ordinary annuity) or at the beginning (annuity due). Here are the results for each type of annuity:

Inputs:	5	6	0	10000	End Mode
	N	I/YR	PV	FV	(Ordinary Annuity)
Output:				-1773.96	

Inputs:	5	6	0	10000	Begin Mode
	N	I/YR	PV	FV	(Annuity Due)
Output:				-1673.55	

Thus, you must save \$1,773.96 per year if you make payments at the *end* of each year, but only \$1,673.55 if the payments begin *immediately*. Note that the required payment for the annuity due is the ordinary annuity payment divided by $(1 + I)$: $\$1,773.96 / 1.06 = \$1,673.55$.

Spreadsheets can also be used to find annuity payments. To find end-of-year payments, we use the PMT function: **=PMT(I,N,PV,FV)**. In our example, this is **=PMT(0.06,5,0,10000) = \$1,773.96**. For the payments of an annuity due (payments at the beginning of the period), the same formula is used, except we must specify the type of annuity, with Type = 1 indicating an annuity due: **=PMT(I,N,PV,FV,Type) = PMT(0.06,5,0,10000,1) = \$1,673.55**.



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

Finding the Number of Periods, N

Suppose you decide to make end-of-year deposits, but you can only save \$1,200 per year. Again assuming that you would earn 6%, how long would it take you to reach your \$10,000 goal? Here is the calculator setup:

Inputs:		6	0	-1200	10000	End Mode
	N	I/YR	PV	PMT	FV	
Output:	6.96					

With these smaller deposits, it would take 6.96 years to reach the \$10,000 target. If you began the deposits immediately, then you would have an annuity due and N would be a bit less, 6.63 years.

In *Excel*, you can use the NPER function: **=NPER(I,PMT,PV,FV)**. For this example, the function is **=NPER(0.06,-1200,0,10000) = 6.96**.



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

Finding the Interest Rate, I

Now suppose you can only save \$1,200 annually, but you still want to have the \$10,000 in five years. What rate of return would enable you to achieve your goal? Here is the calculator setup:

Inputs:	5	0	-1200	10000	End Mode
	N	I/YR	PV	PMT	FV
Output:	25.78				

You would need to earn a whopping 25.78%! About the only possible (but risky) way to get such a high return would be to invest in speculative stocks or head to Las Vegas and a casino. Of course, speculative stocks and gambling aren't like making deposits in a bank with a guaranteed rate of return, so there's a good chance you'd end up with nothing. We'd recommend that you change your plans—save more, lower your \$10,000 target, or extend your time horizon. It might be appropriate to seek a somewhat higher return, but trying to earn 25.78% in a 6% market would require taking on more risk than would be prudent.

In *Excel*, you can use the RATE function: **=RATE(N,PMT,PV,FV)**. For this example, the function is **=RATE(3,-1200,0,10000) = 0.2578 = 25.78 percent**.

SELF-TEST

Suppose you inherited \$100,000 and invested it at 7% per year. How much could you withdraw at the end of each of the next 10 years? How would your answer change if you made withdrawals at the *beginning* of each year? (**\$14,237.75; \$13,306.31**)

If you had \$100,000 that was invested at 7% and you wanted to withdraw \$10,000 at the end of each year, how long would your funds last? How long would they last if you earned 0%? How long would they last if you earned the 7% but limited your withdrawal to \$7,000 per year? (**17.8 years; 10 years; forever**)

Your rich uncle named you as the beneficiary of his life insurance policy. The insurance company gives you a choice of \$100,000 today or a 12-year annuity of \$12,000 at the end of each year. What rate of return is the insurance company offering? (**6.11%**)

Assume that you just inherited an annuity that will pay you \$10,000 per year for 10 years, with the first payment being made today. A friend of your mother offers to give you \$60,000 for the annuity. If you sell it, what rate of return will your mother's friend earn on the investment? If you think a "fair" return would be 6%, how much should you ask for the annuity? (**13.70%; \$78,016.92**)

2.11 Perpetuities

In the last section we dealt with annuities whose payments continue for a specific number of periods—for example, \$100 per year for 10 years. However, some securities promise to make payments forever. For example, in 1749 the British government issued some bonds whose proceeds were used to pay off other British bonds, and since this action consolidated the government's debt, the new bonds were called consols. Since **consols** promise to pay interest perpetually, they are perpetuities. The interest rate on the consols was 2.5%, so a bond with a face value of \$1,000 would pay \$25 per year in perpetuity.⁸

⁸The consols actually pay interest in pounds, but we discuss them in dollar terms for simplicity.

A **perpetuity** is simply an annuity with an extended life. Since the payments go on forever, you can't apply the step-by-step approach. However, it's easy to find the PV of a perpetuity with the following formula:⁹

$$\text{PV of a perpetuity} = \frac{\text{PMT}}{I}. \quad (2-9)$$

Now we can use Equation 2-9 to find the value of a British consol with a face value of \$1,000 that pays \$25 per year in perpetuity. The answer depends on the interest rate. In 1888, the "going rate" as established in the financial marketplace was 2.5%, so at that time the consol's value was \$1,000:

$$\text{Consol value}_{1888} = \$25/0.025 = \$1,000.$$

In 2006, 118 years later, the annual payment was still \$25, but the going interest rate had risen to 5.2%, causing the consol's value to fall to \$480.77:

$$\text{Consol value}_{2006} = \$25/0.052 = \$480.77.$$

Note, though, that if interest rates decline in the future, say to 2%, the value of the consol will rise:

$$\text{Consol value if rates decline to 2\%} = \$25/0.02 = \$1,250.00.$$

These examples demonstrate an important point: *When interest rates change, the prices of outstanding bonds also change. Bond prices decline if rates rise and increase if rates fall.* We will discuss this point in more detail in Chapter 5, where we cover bonds in depth.

SELF-TEST

What is the present value of a perpetuity that pays \$1,000 per year, beginning 1 year from now, if the appropriate interest rate is 5%? What would the value be if the annuity began its payments immediately? (\$20,000, \$21,000). (Hint: Just add the \$1,000 to be received immediately to the value of the annuity.) What happens to the value of a bond if interest rates go up? If rates go down?

2.12 Uneven Cash Flows

The definition of an annuity includes the term *constant payment*—in other words, annuities involve payments that are equal in every period. Although many financial decisions do involve constant payments, many others involve **nonconstant**, or **uneven, cash flows**. For example, the dividends on common stocks typically increase over time, and investments in capital equipment almost always generate uneven cash flows. Throughout the book, we reserve the term **payment (PMT)** for annuities with their equal payments in each period and use the term **cash flow (CF_t)** to denote uneven cash flows, where the *t* designates the period in which the cash flow occurs.

There are two important classes of uneven cash flows: (1) a stream that consists of a series of annuity payments plus an additional final lump sum and (2) all other uneven streams. Bonds represent the best example of the first type, while

⁹See **Web Extension 2A** at the textbook's Web site for a derivation of the perpetuity formula.

stocks and capital investments illustrate the other type. Here are numerical examples of the two types of flows:

1. Annuity plus additional final payment:

Periods	0	1	2	3	4	5
Cash flows	\$0	\$100	\$100	\$100	\$100	\$1,100
						\$1,000
						<u>\$1,100</u>

2. Irregular cash flows:

Periods	0	1	2	3	4	5
Cash flows	\$0	\$100	\$300	\$300	\$300	\$500

We can find the PV of either stream by using Equation 2-10 and following the step-by-step procedure, where we discount each cash flow and then sum them to find the PV of the stream:

$$PV = \frac{CF_1}{(1 + I)^1} + \frac{CF_2}{(1 + I)^2} + \dots + \frac{CF_N}{(1 + I)^N} = \sum_{t=1}^N \frac{CF_t}{(1 + I)^t} \quad (2-10)$$

If we did this, we would find the PV of Stream 1 to be \$927.90 and the PV of Stream 2 to be \$1,016.35.

The step-by-step procedure is straightforward, but if we have a large number of cash flows it is time-consuming. However, financial calculators and spreadsheets speed up the process considerably.

First, consider Stream 1, and notice that it is a 5-year, 12%, ordinary annuity plus a final payment of \$1,000. We could find the PV of the annuity, then find the PV of the final payment, and then sum them to get the PV of the stream. However, financial calculators do this in one simple step—use the five TVM keys, enter the data as shown below, and then press the PV key to get the answer, \$927.90.

Inputs:	5	12	100	1000
	N	I/YR	PMT	FV
Output:			-927.90	

Similarly, we could use the PV function in *Excel*: **=PV(I,N,PMT,FV)**. This is similar to our previous solutions for an annuity, except we now have a nonzero value for FV. Applying the PV function, we get **=PV(0.12,5,100,1000) = -\$927.90**.

Now consider the second uneven stream shown in Figure 2-7. The top section shows the time line and an application of the step-by-step approach.

Because the cash flows do not represent an annuity, you can't use the annuity feature on your financial calculator to find the present value. However, your financial calculator does have a feature that allows you to find the present value. First, you input the individual cash flows, in chronological order, into the cash flow register.¹⁰ Cash flows are usually designated CF₀, CF₁, CF₂, CF₃, and so on.

¹⁰We cover the calculator mechanics in the tutorial, and we discuss the process in more detail in Chapter 11, where we use the NPV calculation to analyze proposed projects. If you don't know how to do the calculation with your calculator, it will be worthwhile to go to our tutorial or your calculator manual, learn the steps, and be sure you can make this calculation. You will have to learn to do it eventually, and now is a good time.



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

Figure 2-7

PV of an Uneven Cash Flow Stream

	A	B	C	D	E	F	G
358	Interest rate	= I =	12%				
359							
360	Periods:	0	1	2	3	4	5
361							
362	CF Time Line:	\$0	\$100	\$300	\$300	\$300	\$500
363							
364		PV of CFs					
365		\$89.29					
366		239.16					
367		213.53					
368		190.66					
369		283.71					
370		\$1,016.35	= PV of cash flow stream = value of the asset				
371							
372	Excel Function Approach:		Fixed inputs:	NPV =	=NPV(0.12,100,300,300,300,500)		1,016.35
373			Cell references:	NPV =	=NPV(C358,C362:G362)		1,016.35

Next, you enter the interest rate, I. At this point, you have substituted in all the known values of Equation 2-10, so you only need to press the NPV key to find the present value of the stream. The calculator has been programmed to find the PV of each cash flow and then to sum these values to find the PV of the entire stream. To input the cash flows for this problem, enter 0 (because $CF_0 = 0$), 100, 300, 300, 300, and 500 in that order into the cash flow register, enter $I = 12$, and then press NPV to obtain the answer, \$1,016.35.

Two points should be noted. First, when dealing with the cash flow register, the calculator uses the term “NPV” rather than “PV.” The N stands for “net,” so NPV is the abbreviation for “Net Present Value,” which is simply the net present value of a series of positive and negative cash flows, including the cash flow at time zero.

The second point to note is that repeated cash flows with identical values can be entered into the cash flow register more efficiently by using the Nj key. In this illustration, you would enter $CF_0 = 0$, $CF_1 = 100$, $CF_2 = 300$, Nj = 3 (which tells the calculator that the 300 occurs 3 times), and $CF_5 = 500$.¹¹ Then enter $I = 12$, press the NPV key, and 1,016.35 will appear in the display. Also, note that amounts entered into the cash flow register remain in the register until they are cleared. Thus, if you had previously worked a problem with eight cash flows, and then moved to a problem with only four cash flows, the calculator would simply add the cash flows from the second problem to those of the first problem. Therefore, you must be sure to clear the cash flow register before starting a new problem.

Spreadsheets are especially useful for solving problems with uneven cash flows. Just as with a financial calculator, you must enter the cash flows in the spreadsheet, as shown in Figure 2-7. To find the PV of these cash flows with *Excel*, you can use the NPV function. Put your cursor on Cell G373, click the function wizard, click Financial, scroll down to NPV, and click OK to get the dialog box.

¹¹On some calculators, instead of entering $CF_5 = 500$, you enter $CF_3 = 500$, because this is the next *different* cash flow.

Then enter C358 or 0.12 for Rate and the range of cells containing the cash flows, C362:G362, for Value 1. Be very careful when entering the range of cash flows. With a financial calculator, you begin by entering the Time 0 cash flow. With *Excel*, you do *not* include the Time 0 cash flow; instead, you begin with the Year 1 cash flow. Now, when you click OK, you get the PV of the stream, \$1,016.35. Note that you can use the PV function if the payments are constant but you must use the NPV function if the cash flows are not constant. Note too that one of the advantages of spreadsheets over financial calculators is that you can see the cash flows, which makes it easy to spot any typing errors.

SELF-TEST

Could you use Equation 2-3, once for each cash flow, to find the PV of an uneven stream of cash flows? What is the present value of a 5-year ordinary annuity of \$100 plus an additional \$500 at the end of Year 5 if the interest rate is 6%? How would the PV change if the \$100 payments occurred in Years 1 through 10 and the \$500 came at the end of Year 10? (**\$794.87**; **\$1,015.21**)

What is the present value of the following uneven cash flow stream: \$0 at Time 0, \$100 in Year 1 (or at Time 1), \$200 in Year 2, \$0 in Year 3, and \$400 in Year 4 if the interest rate is 8%? (**\$558.07**)

Would a typical common stock provide cash flows more like an annuity or more like an uneven cash flow stream?

2.13 Future Value of an Uneven Cash Flow Stream

The future value of an uneven cash flow stream (sometimes called the **terminal value**) is found by compounding each payment to the end of the stream and then summing the future values:

$$\begin{aligned} \text{FV} &= \text{CF}_0(1 + I)^N + \text{CF}_1(1 + I)^{N-1} + \text{CF}_2(1 + I)^{N-2} + \dots + \text{CF}_{N-1}(1 + I) + \text{CF}_N \\ &= \sum_{t=0}^N \text{CF}_t(1 + I)^{N-t} \end{aligned} \quad (2-11)$$

The future value of our illustrative uneven cash flow stream is \$1,791.15, as shown in Figure 2-8.

Some financial calculators have a net future value (NFV) key which, after the cash flows and interest rate have been entered, can be used to obtain the future value of an uneven cash flow stream. Even if your calculator doesn't have the NFV feature, you can use the cash flow stream's net present value to find its net future value: $\text{NFV} = \text{NPV}(1 + I)^N$. Thus, in our example, you could find the PV of the stream, then find the FV of that PV, compounded for N periods at I%. In the illustrative problem, find $\text{PV} = 1,016.35$ using the cash flow register and $I = 12$. Then enter $N = 5$, $I = 12$, $\text{PV} = -1016.35$, and $\text{PMT} = 0$. Press FV to find $\text{FV} = 1,791.15$, which equals the NFV shown on the time line in Figure 2-8. A similar procedure can be used in *Excel*: First find the NPV, then find its FV. See *FM12 Ch 02 Tool Kit.xls* for details.



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See *FM12 Ch 02 Tool Kit.xls* for all calculations.

SELF-TEST

What is the future value of this cash flow stream: \$100 at the end of 1 year, \$150 due after 2 years, and \$300 due after 3 years if the appropriate interest rate is 15%? (**\$604.75**)

Figure 2-8

FV of an Uneven Cash Flow Stream

	A	B	C	D	E	F	G
383	Interest rate	= I =	12%				
384							
385	Periods:	0	1	2	3	4	5
386							
387	CF Time Line:	\$0	\$100	\$300	\$300	\$300	\$500
388							\$500.00
389							336.00
390							376.32
391							421.48
392							157.35
393							0.00
394						FV of cash flow stream =	\$1,791.15
395							
396							
397	Excel Function Approach:	First find NPV:	NPV =	=NPV(C383,C387:G387)			1,016.35
398		Then compound NPV for 5 years:	NFV =	=FV(C383,G385,-G397)			1,791.15

2.14 Solving for I with Uneven Cash Flows

Before financial calculators and spreadsheets existed, it was *extremely difficult* to find I if the cash flows were uneven. With spreadsheets and financial calculators, though, it's relatively easy to find I. If you have an annuity plus a final lump sum, you can input values for N, PV, PMT, and FV into the calculator's TVM registers and then press the I/YR key. Here is the setup for Stream 1 from Section 2.12, assuming we must pay \$927.90 to buy the asset. The rate of return on the \$927.90 investment is 12%, as shown below:

Inputs:	5	-927.90	100	1000
	N	I/YR	PV	PMT
Output:		12.00		

Finding the interest rate for an uneven cash flow stream is a bit more complicated. Figure 2-9 shows Stream 2 from Section 2.12, assuming CF_0 is $-\$1,000$. First, note that there is no simple procedure for finding the rate of return—finding the rate requires a trial-and-error process, which means that one really needs a financial calculator or a spreadsheet. With a calculator, we would enter the CFs into the cash flow register and then press the IRR key to get the answer. IRR stands for “internal rate of return,” and it is the rate of return the investment provides. The investment is the cash flow at Time 0, and it must be entered as a negative number. When we enter those cash flows in the calculator's



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

Figure 2-9

IRR of an Uneven Cash Flow Stream

	A	B	C	D	E	F	G
421	Periods:	0	1	2	3	4	5
422		----- ----- ----- ----- -----					
423	CF Time Line:	-\$1,000	\$100	\$300	\$300	\$300	\$500
424							
425	Excel Function Approach:	Cell references:	IRR =	=IRR(B423:G423)			12.55%

cash flow register and press the IRR key, we get the rate of return on the \$1,000 investment, 12.55%.

You would get the same answer using *Excel's* IRR function, as shown in Figure 2-9. Notice that with the IRR function, you must include all cash flows, including the Time 0 cash flow; with the NPV function, you do not include the Time 0 cash flow.

SELF-TEST

An investment costs \$465 and is expected to produce cash flows of \$100 at the end of each of the next 4 years, then an extra lump sum payment of \$200 at the end of the 4th year. What is the expected rate of return on this investment? (9.05%)

An investment costs \$465 and is expected to produce cash flows of \$100 at the end of Year 1, \$200 at the end of Year 2, and \$300 at the end of Year 3. What is the expected rate of return on this investment? (11.71%)

2.15 Semiannual and Other Compounding Periods

In almost all of our examples thus far, we have assumed that interest is compounded once a year, or annually. This is called **annual compounding**. Suppose, however, that you put \$100 into a bank that states it pays a 6% annual interest rate but that interest is credited each 6 months. This is called **semiannual compounding**. How much would you have accumulated at the end of 1 year, 2 years, or some other period under semiannual compounding? Note that virtually all bonds pay interest semiannually, most stocks pay dividends quarterly, and most mortgages, student loans, and auto loans require monthly payments. Therefore, it is essential that you understand how to deal with nonannual compounding.

Types of Interest Rates

Compounding involves three types of interest rates: nominal rates, I_{NOM} ; periodic rates, I_{PER} ; and effective annual rates, EAR or EFF%.

Nominal, or Quoted, Rate, I_{NOM} ¹² This is the rate that is quoted by banks, brokers, and other financial institutions. So, if you talk with a banker, broker, mortgage lender, auto finance company, or student loan officer about rates, the nominal rate is the one he or she will normally quote you. However, to be meaningful, the quoted nominal rate must also include the number of compounding periods per year. For example, a bank might offer 6%, compounded quarterly, on CDs; or a mutual fund might offer 5%, compounded monthly, on its money market account.

¹²The term *nominal rate* as it is used here has a different meaning than the way it was used in Chapter 1. There, nominal interest rates referred to stated market rates as opposed to real (zero inflation) rates. In this chapter, the term *nominal rate* means the stated, or quoted, annual rate as opposed to the effective annual rate, which we explain later. In both cases, though, *nominal* means *stated*, or *quoted*, as opposed to some adjusted rate.

The nominal rate on loans to consumers is also called the **Annual Percentage Rate (APR)**. For example, if a credit card issuer quotes an annual rate of 18%, this is the APR.

Note that the nominal rate is never shown on a time line, and it is never used as an input in a financial calculator, unless compounding occurs only once a year. If more frequent compounding occurs, you should use the periodic rate as discussed below.

Periodic Rate, I_{PER} This is the rate charged by a lender or paid by a borrower each period. It can be a rate per year, per 6-month period, per quarter, per month, per day, or per any other time interval. For example, a bank might charge 1.5% per month on its credit card loans, or a finance company might charge 3% per quarter on installment loans. We find the periodic rate as follows:

$$\text{Periodic rate, } I_{PER} = I_{NOM}/M, \quad (2-12)$$

which implies that

$$\text{Nominal annual rate} = I_{NOM} = (\text{Periodic rate})(M). \quad (2-13)$$

Here I_{NOM} is the nominal annual rate and M is the number of compounding periods per year. To illustrate, consider a finance company loan at 3% per quarter:

$$\text{Nominal annual rate} = I_{NOM} = (\text{Periodic rate})(M) = (3\%)(4) = 12\%,$$

or

$$\text{Periodic rate} = I_{NOM}/M = 12\%/4 = 3\% \text{ per quarter.}$$

If there is only one payment per year, or if interest is added only once a year, then $M = 1$, and the periodic rate is equal to the nominal rate.

*The periodic rate is the rate that is generally shown on time lines and used in calculations.*¹³ To illustrate use of the periodic rate, suppose you invest \$100 in an account that pays a nominal rate of 12%, compounded quarterly. How much would you have after 2 years?

For compounding more frequently than annually, we use the following modification of Equation 2-1:

$$FV_N = PV(1 + I_{PER})^{\text{Number of periods}} = PV\left(1 + \frac{I_{NOM}}{M}\right)^{MN}. \quad (2-14)$$

For the example of \$100 compounded quarterly at an annual nominal rate of 12% for 2 years, the time line is



¹³The only exception is in situations where (1) annuities are involved and (2) the payment periods do not correspond to the compounding periods. If an annuity is involved and if its payment periods do not correspond to the compounding periods—for example, if you are making quarterly payments into a bank account to build up a specified future sum, but the bank pays interest on a daily basis—then the calculations are more complicated. For such problems, one can proceed in two alternative ways. (1) Determine the periodic (daily) interest rate by dividing the nominal rate by 365 (or 360 if the bank uses a 360-day year), then compound each payment over the exact number of days from the payment date to the terminal point, and then sum the compounded payments to find the future value of the annuity. This is what would generally be done in the real world, because with a computer it would be a simple process. (2) Calculate the EAR, as defined later in this section, based on daily compounding, then find the corresponding nominal rate based on quarterly compounding (because the annuity payments are made quarterly), then find the quarterly periodic rate, and use that rate with standard annuity procedures. The second procedure is faster with a calculator, but hard to explain and generally not used in practice given the ready availability of computers.

Using Equation 2-14,

$$\begin{aligned} FV &= \$100(1 + 0.03)^8 \\ &= \$126.68. \end{aligned}$$

With a financial calculator, we work the problem with inputs $N = 2 \times 4 = 8$, $I = 12/4 = 3$, $PV = -100$, and $PMT = 0$. The result is $FV = \$126.68$.

Inputs:	8	3	-100	0	
	N	I/YR	PV	PMT	FV
Output:					126.68

Most financial calculators have a feature that allows you to set the number of payments per year and then use the nominal annual interest rate. However, in our experience, students make fewer errors when using the periodic rate with their calculators set for one payment per year (i.e., per period), so this is what we recommend.

Effective (or Equivalent) Annual Rate (EAR or EFF%) This is the annual rate that produces the same result as if we had compounded at a given periodic rate M times per year. The EAR, also called EFF% (for effective percentage), is found as follows:

$$\text{Effective annual rate (EFF\%)} = \left(1 + \frac{I_{\text{NOM}}}{M}\right)^M - 1.0. \quad (2-15)$$

You could also use the interest conversion feature of a financial calculator.¹⁴

In the EAR equation, I_{NOM}/M is the periodic rate, and M is the number of periods per year. For example, suppose you could borrow using either a credit card that charges 1% per month or a bank loan with a 12% quoted nominal interest rate that is compounded quarterly. Which should you choose? To answer this question, the cost rate of each alternative must be expressed as an EAR:

$$\begin{aligned} \text{Credit card loan: EFF\%} &= (1 + 0.01)^{12} - 1.0 = (1.01)^{12} - 1.0 \\ &= 1.126825 - 1.0 = 0.126825 = 12.6825\%. \end{aligned}$$

$$\begin{aligned} \text{Bank loan: EFF\%} &= (1 + 0.03)^4 - 1.0 = (1.03)^4 - 1.0 \\ &= 1.125509 - 1.0 = 0.125509 = 12.5509\%. \end{aligned}$$

Thus, the credit card loan is slightly more costly than the bank loan. This result should have been intuitive to you—both loans have the same 12% nominal rate, yet you would have to make monthly payments on the credit card versus quarterly payments under the bank loan.

The EFF% rate is not used in calculations. However, it should be used to compare the effective cost or rate of return on loans or investments when payment periods differ, as in the credit card versus bank loan example.



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

¹⁴Most financial calculators are programmed to find the EFF% or, given the EFF%, to find the nominal rate. This is called “interest rate conversion.” You enter the nominal rate and the number of compounding periods per year and then press the EFF% key to find the effective annual rate. However, we generally use Equation 2-15 because it’s as easy to use as the interest conversion feature is, and the equation reminds us of what we are really doing. If you use the interest rate conversion feature on your calculator, don’t forget to reset your calculator settings. Interest conversion is discussed in the tutorials.

The Result of Frequent Compounding

Suppose you plan to invest \$100 for 5 years at a nominal annual rate of 10%. What will happen to the future value of your investment if interest is compounded more frequently than once a year? Because interest will be earned on interest more often, you might expect the future value to increase as the frequency of compounding increases. Similarly, you might also expect the effective annual rate to increase with more frequent compounding. As Table 2-1 shows, you would be correct—the future value and EAR do in fact increase as the frequency of compounding increases. Notice that the biggest increases in FV and EAR occur when compounding goes from annual to semiannual, and that moving from monthly to daily compounding has a relatively small impact. Although Table 2-1 shows daily compounding as the smallest interval, it is possible to compound even more frequently. At the limit, one can have **continuous compounding**. This is explained in *Web Extension 2B* at the textbook's Web site.

Table 2-1

The Impact of Frequent Compounding

Frequency of Compounding	Nominal Annual Rate	Effective Annual Rate (EFF%) ^a	Future Value of \$100 Invested for 5 Years ^b
Annual	10.00%	10.000%	\$161.05
Semiannual	10.00	10.250	162.89
Quarterly	10.00	10.381	163.86
Monthly	10.00	10.471	164.53
Daily ^c	10.00	10.516	164.86

^aThe EAR is calculated using Equation 2-14.

^bThe future value is calculated using Equation 2-13.

^cThe daily calculations assume 365 days per year.

SELF-TEST

Would you rather invest in an account that pays 7% with annual compounding or 7% with monthly compounding? Would you rather borrow at 7% and make annual or monthly payments? Why?

What is the *future value* of \$100 after 3 years if the appropriate interest rate is 8%, compounded annually? Compounded monthly? (\$125.97; \$127.02)

What is the *present value* of \$100 due in 3 years if the appropriate interest rate is 8%, compounded annually? Compounded monthly? (\$79.38; \$78.73)

Define the terms "Annual Percentage Rate, or APR," "effective annual rate, or EFF%," and "nominal interest rate, I_{NOM} ."

A bank pays 5% with daily compounding on its savings accounts. Should it advertise the nominal or effective rate if it is seeking to attract new deposits?

Credit card issuers must by law print their Annual Percentage Rate (APR) on their monthly statements. A common APR is 18%, with interest paid monthly. What is the EFF% on such a loan? (EFF% = $(1 + 0.18/12)^{12} - 1 = 0.1956 = 19.56\%$.)

Some years ago banks didn't have to reveal the rate they charged on credit cards. Then Congress passed a "truth in lending" law that required them to publish their APR. Is the APR really the "most truthful" rate, or would the EFF% be "more truthful?"

Using the Internet for Personal Financial Planning



People continually face important financial decisions that require an understanding of the time value of money. Should we buy or lease a car? How much and how soon do we need to save for our children's education? What size house can we afford? Should we refinance our home mortgage? How much must we save in order to retire comfortably?

The answers to these questions are often complicated, and they depend on a number of factors, such as housing and education costs, interest rates, inflation, expected family income, and stock market returns. Hopefully, after completing this chapter, you

will have a better idea of how to answer such questions. Moreover, there are a number of online resources available to help with financial planning.

A good place to start is <http://www.smartmoney.com>. *Smartmoney* is a personal finance magazine produced by the publishers of *The Wall Street Journal*. If you go to *Smartmoney's* Web site you will find a section entitled "Tools and Maps." This section has a number of financial calculators, spreadsheets, and descriptive materials that cover a wide range of personal finance issues.

2.16 Fractional Time Periods

Thus far we have assumed that payments occur at either the beginning or the end of periods, but not *within* periods. However, we often encounter situations that require compounding or discounting over fractional periods. For example, suppose you deposited \$100 in a bank that pays a nominal rate of 10% but adds interest daily, based on a 365-day year. How much would you have after 9 months? The answer is \$107.79, found as follows:¹⁵

$$\text{Periodic rate} = I_{\text{PER}} = 0.10/365 = 0.000273973 \text{ per day.}$$

$$\begin{aligned} \text{Number of days} &= (9/12)(365) = 0.75(365) \\ &= 273.75 \text{ rounded to } 274. \end{aligned}$$

$$\text{Ending amount} = \$100(1.000273973)^{274} = \$107.79.$$

Now suppose you borrow \$100 from a bank whose nominal rate is 10% per year "simple interest," which means that interest is not earned on interest. If the loan is outstanding for 274 days, how much interest would you have to pay? Here we would calculate a daily interest rate, I_{PER} , as above, but multiply it by 274 rather than use the 274 as an exponent:

$$\text{Interest owed} = \$100(0.000273973)(274) = \$7.51 \text{ interest.}$$

You would owe the bank a total of \$107.51 after 274 days. This is the procedure most banks actually use to calculate interest on loans, except that they require borrowers to pay the interest on a monthly basis rather than after 274 days.

SELF-TEST

Suppose a company borrowed \$1 million at a rate of 9%, simple interest, with interest paid at the end of each month. The bank uses a 360-day year. How much interest would the firm have to pay in a 30-day month? What would the interest be if the bank used a 365-day year? $[(0.09/360)(30)](\$1,000,000) = \$7,500$ interest for the month. For the 365-day year, $(0.09/365)(30)(\$1,000,000) = \$7,397.26$ of interest. The use of a 360-day year raises the interest cost by \$102.74. That's why banks like to use it on loans.]

(Continued)

¹⁵Bank loan contracts specifically state whether they are based on a 360- or a 365-day year. If a 360-day year is used, then the daily rate is higher, so the effective rate is also higher. Here we assumed a 365-day year. Also, note that in real-world calculations, banks' computers have built-in calendars, so they can calculate the exact number of days, taking account of 30-day, 31-day, and 28- or 29-day months.

SELF-TEST

Suppose you deposited \$1,000 in a credit union that pays 7% with daily compounding and a 365-day year. What is the EFF%, and how much could you withdraw after 7 months, assuming this is 7/12 of a year? [EFF% = $(1 + 0.07/365)^{365} - 1 = 0.07250098 = 7.250098\%$. Thus, your account would grow from \$1,000 to $\$1,000(1.07250098)^{0.583333} = \$1,041.67$, and you could withdraw that amount.]

2.17 Amortized Loans

An important application of compound interest involves loans that are paid off in installments over time. Included are automobile loans, home mortgage loans, student loans, and many business loans. A loan that is to be repaid in equal amounts on a monthly, quarterly, or annual basis is called an **amortized loan**.¹⁶

Table 2-2 illustrates the amortization process. A homeowner borrows \$100,000 on a mortgage loan, and the loan is to be repaid in 5 equal payments at the end of each of the next 5 years.¹⁷ The lender charges 6% on the balance at the beginning of each year.

Our first task is to determine the payment the homeowner must make each year. Here's a picture of the situation:



The payments must be such that the sum of their PVs equals \$100,000:

$$\$100,000 = \frac{\text{PMT}}{(1.06)^1} + \frac{\text{PMT}}{(1.06)^2} + \frac{\text{PMT}}{(1.06)^3} + \frac{\text{PMT}}{(1.06)^4} + \frac{\text{PMT}}{(1.06)^5} = \sum_{t=1}^5 \frac{\text{PMT}}{(1.06)^t}$$

It is possible to use the annuity formula in Equation 2-7, but it is much easier to use a financial calculator or spreadsheet. With a financial calculator, we insert values into a calculator as shown below to get the required payments, \$23,739.64:

Inputs:	5	6	100000		0
	N	I/YR	PV	PMT	FV
Output:				-23739.64	

Therefore, the borrower must pay the lender \$23,739.64 per year for the next 5 years.

With *Excel*, you would use the PMT function: `=PMT(I,N,PV,FV) = PMT(0.06,5,100000,0) = -23,739.64`.

Each payment will consist of two parts—interest and repayment of principal. This breakdown is shown on an **amortization schedule** such as the one in Table 2-2. The interest component is relatively high in the first year, but it declines as the loan balance decreases. For tax purposes, the borrower would deduct the interest component while the lender would report the same amount as taxable income.

¹⁶The word *amortized* comes from the Latin *mors*, meaning “death,” so an amortized loan is one that is “killed off” over time.

¹⁷Most mortgage loans call for monthly payments over 10 to 30 years, but we use a shorter period to simplify the example.



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See **FM12 Ch 02 Tool Kit.xls** for all calculations.

Table 2-2

Loan Amortization Schedule, \$100,000 at 6% for 5 Years

Year	Beginning Amount (1)	Payment (2)	Interest ^a (3)	Repayment of Principal ^b (2) – (3) = (4)	Ending Balance (1) – (4) = (5)
1	\$100,000.00	\$23,739.64	\$6,000.00	\$17,739.64	\$82,260.36
2	\$82,260.36	\$23,739.64	\$4,935.62	\$18,804.02	\$63,456.34
3	\$63,456.34	\$23,739.64	\$3,807.38	\$19,932.26	\$43,524.08
4	\$43,524.08	\$23,739.64	\$2,611.44	\$21,128.20	\$22,395.89
5	\$22,395.89	\$23,739.64	\$1,343.75	\$22,395.89	\$0.00

^aInterest in each period is calculated by multiplying the loan balance at the beginning of the year by the interest rate. Therefore, interest in Year 1 is $\$100,000(0.06) = \$6,000$; in Year 2 it is $\$82,260.36(0.06) = \$4,935.62$; and so on.

^bRepayment of principal is equal to the payment of \$23,739.64 minus the interest charged for the year.

SELF-TEST

Suppose you borrowed \$30,000 on a student loan at a rate of 8% and now must repay it in 3 equal installments at the end of each of the next 3 years. How large would your payments be, how much of the first payment would represent interest, how much would be principal, and what would your ending balance be after the first year? (PMT = \$11,641.01; Interest = \$2,400; Principal = \$9,241.01; Balance at end of Year 1 = \$20,758.99)

2.18 Growing Annuities

Normally, an annuity is defined as a series of *constant* payments to be received over a specified number of periods. However, the term **growing annuity** is used to describe a series of payments that grow at a constant rate.

Example 1: Finding a Constant Real Income. Growing annuities are often used in the area of financial planning, where a prospective retiree wants to determine the maximum constant *real*, or *inflation-adjusted*, withdrawals he or she can make over a specified number of years. For example, suppose a 65-year-old is contemplating retirement, expects to live for another 20 years, has a \$1 million nest egg, expects to earn 8% on his or her investments, expects inflation to average 3% per year, and wants to withdraw a constant *real* amount annually over the next remaining 20 years. If the first withdrawal is to be made today, what is the amount of this initial withdrawal?

This problem can be solved in three ways. (1) Set up a spreadsheet model that is similar to an amortization table, where the account earns 8% per year, withdrawals rise at the 3% inflation rate, and *Excel's* Goal Seek function is used to find the initial inflation-adjusted withdrawal, and a zero balance remains at the end of the 20th year. (2) Use a financial calculator, where we first calculate the real rate of return, adjusted for inflation, and use it for I/YR when finding the payment for an annuity due. (3) Use a relatively complicated and obtuse formula to find this same amount.¹⁸ We illustrate the first two procedures in the chapter

model, *FM12 Ch 02 Tool Kit.xls*. The financial calculator approach is easiest to use, but the spreadsheet model provides the clearest picture of what is happening. It shows the value of the retirement portfolio, the earning, and each withdrawal over the 20-year planning horizon. Also, the spreadsheet model creates graphs that make it easy to explain things to people who are planning their financial futures.

To implement the calculator approach, we first find the expected real, or inflation adjusted, rate of return, where r_r is the real rate of return and r_{NOM} the nominal rate of return:

$$\begin{aligned}\text{Real rate} = r_r &= [(1 + r_{\text{NOM}})/(1 + \text{Inflation})] - 1.0 \\ &= [1.08/1.03] - 1.0 = 0.048543689 = 4.8543689\%.\end{aligned}\quad (2-16)$$

Next, we set the calculator to Begin Mode, after which we input $N = 20$, $I = \text{real rate} = 4.8543689$, $PV = -1,000,000$, and $FV = 0$, then press PMT to get \$75,585.53. This is the value at Time 0, today, and the initial withdrawal will increase at the inflation rate, 3%. With this initial withdrawal, the retiree will have a constant real income over the next 20 years.

In our example we assumed that the first withdrawal would be made immediately. The procedure would be slightly different if we wanted to make end-of-year withdrawals. First, we would find the real rate the same way and enter the same inputs into the calculator as shown above. Second, we would the calculator set to End Mode. The calculated PMT would be \$79,254.73. However, that value is in beginning-of-year terms, and since inflation of 3% will occur during the year, we must make the following adjustment to find the inflation-adjusted initial payment:

$$\begin{aligned}\text{Initial withdrawal} &= \$79,254.73(1 + \text{Inflation}) \\ &= \$79,254.73(1.03) \\ &= \$81,632.38.\end{aligned}$$

The first withdrawal, at the end of the year, would thus be \$81,632.38, it would grow by 3% per year, and after the 20th withdrawal (at the end of the 20th year) the balance in the retirement fund would be zero.

The end-of-year payment is also analyzed in *FM12 Ch 02 Tool Kit.xls*. There we set up an amortization table that shows the beginning balance, the annual withdrawals, the annual earnings, and the ending balance for each of the 20 years. This analysis confirms the \$81,632.38 initial withdrawal found above.

Example 2: Initial Deposit to Accumulate a Future Sum. As another example of a growing annuity, suppose you need to accumulate \$100,000 in 10 years. You plan to make a deposit now, at Time 0, and then to make 9 more deposits at the beginning of the following 9 years, for a total of 10 deposits. The bank pays 6% interest, and you expect to increase your initial deposit amount by the 2% inflation rate each year. How much would you need to deposit initially? First, we calculate the real rate:

$$\text{Real rate} = r_r = [1.06/1.02] - 1.0 = 0.0392157 = 3.92157\%.$$

¹⁸For example, the formula used to find the payment for a growing ordinary annuity is shown below. If g = annuity growth rate and l = rate of return on investment, then

$$\text{PVIF of a growing annuity} = \text{PVIFGA} = [1 - [(1 + g)/(1 + l)]^N]/[(l - g)/(1 + g)].$$

$$\text{PMT} = \text{PV}/\text{PVIFGA}.$$



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See *FM12 Ch 02 Tool Kit.xls* for all calculations.

Next, since inflation is expected to be 2% per year, then in 10 years the target \$100,000 will have a purchasing power of:

$$\$100,000/(1 + 0.02)^{10} = \$82,034.83.$$

Now we can find the size of the required initial payment by setting a financial calculator to the “BEG” mode and then inputting $N = 10$, $I/YR = 3.92157$, $PV = 0$, and $FV = 82,034.83$. Then, when we press the PMT key, we get $PMT = -6,598.87$. Thus, a deposit of \$6,598.87 made at time zero and growing by 2 percent per year will accumulate to \$100,000 by Year 10 if the interest rate is 6%. Again, this result is confirmed in the chapter model. The key to this analysis is to express I/YR , PV , and PMT in real, inflation-adjusted terms.

SELF-TEST

Differentiate between a “regular” and a growing annuity.

What three methods can be used to deal with growing annuities?

If the nominal interest rate is 10% and the expected inflation rate is 5%, what is the expected real rate of return? (4.7619%)

Summary

Most financial decisions involve situations in which someone pays money at one point in time and receives money at some later time. Dollars paid or received at two different points in time are different, and this difference is recognized and accounted for by *time value of money (TVM) analysis*.

- **Compounding** is the process of determining the **future value (FV)** of a cash flow or a series of cash flows. The compounded amount, or future value, is equal to the beginning amount plus the interest earned.
- Future value of a single payment: $FV_N = PV(1 + I)^N$.
- **Discounting** is the process of finding the **present value (PV)** of a future cash flow or a series of cash flows; discounting is the reciprocal, or reverse, of compounding.

- Present value of a single payment: $PV = \frac{FV_N}{(1 + I)^N}$.

- An **annuity** is defined as a series of equal periodic payments (PMT) for a specified number of periods.
- Future value of an annuity:

$$FVA_N = PMT \left[\frac{(1 + I)^N}{I} - \frac{1}{I} \right].$$

- Present value of an annuity:

$$PVA_N = PMT \left[\frac{1}{I} - \frac{1}{I(1 + I)^N} \right].$$

- An annuity whose payments occur at the *end* of each period is called an **ordinary annuity**. The formulas above are for ordinary annuities.
- If each payment occurs at the beginning of the period rather than at the end, then we have an **annuity due**. The PV of each payment would be larger,

because each payment would be discounted back one year less, so the PV of the annuity would also be larger. Similarly, the FV of the annuity due would also be larger because each payment would be compounded for an extra year. The following formulas can be used to convert the PV and FV of an ordinary annuity to an annuity due:

$$\text{PVA (annuity due)} = \text{PVA of an ordinary annuity} \times (1 + I),$$

$$\text{FVA (annuity due)} = \text{FVA of an ordinary annuity} \times (1 + I).$$

- A **perpetuity** is an annuity with an infinite number of payments.

$$\text{Value of a perpetuity} = \frac{\text{PMT}}{I}.$$

- To find the PV or FV of an uneven series, find the PV or FV of each individual cash flow and then sum them.
- If you know the cash flows and the PV (or FV) of a cash flow stream, you can **determine the interest rate**.
- When compounding occurs more frequently than once a year, the nominal rate must be converted to a periodic rate, and the number of years must be converted to periods.

$$I_{\text{PER}} = \text{Nominal annual rate} / \text{Periods per year}$$

$$\text{Periods} = \text{Years} \times \text{Periods per year}$$

The periodic rate and number of periods would be used for calculations and shown on time lines.

- If comparing the costs of loans that require payments more than once a year, or the rates of return on investments that pay interest more frequently, then the comparisons should be based on **equivalent** (or **effective**) rates of return using this formula:

$$\text{Effective annual rate (EFF\%)} = \left(1 + \frac{I_{\text{NOM}}}{M} \right)^M - 1.0.$$

- The general equation for finding the future value for any number of compounding periods per year is

$$\text{FV}_N = \text{PV}(1 + I_{\text{PER}})^{\text{Number of periods}} = \text{PV} \left(1 + \frac{I_{\text{NOM}}}{M} \right)^{MN}$$

where

$$I_{\text{NOM}} = \text{Nominal quoted interest rate,}$$

$$M = \text{Number of compounding periods per year,}$$

$$N = \text{Number of years.}$$

- An **amortized loan** is one that is paid off in equal payments over a specified period. An **amortization schedule** shows how much of each payment constitutes interest, how much is used to reduce the principal, and the unpaid balance at each point in time.

Questions

- (2-1) Define each of the following terms:
- PV; I; INT; FV_N ; PVA_N ; FVA_N ; PMT; M; I_{NOM}
 - Opportunity cost rate
 - Annuity; lump sum payment; cash flow; uneven cash flow stream
 - Ordinary (deferred) annuity; annuity due
 - Perpetuity; consol
 - Outflow; inflow; time line; terminal value
 - Compounding; discounting
 - Annual, semiannual, quarterly, monthly, and daily compounding
 - Effective annual rate (EAR); nominal (quoted) interest rate; APR; periodic rate
 - Amortization schedule; principal versus interest component of a payment; amortized loan
- (2-2) What is an *opportunity cost rate*? How is this rate used in discounted cash flow analysis, and where is it shown on a time line? Is the opportunity rate a single number that is used in all situations?
- (2-3) An *annuity* is defined as a series of payments of a fixed amount for a specific number of periods. Thus, \$100 a year for 10 years is an annuity, but \$100 in Year 1, \$200 in Year 2, and \$400 in Years 3 through 10 does *not* constitute an annuity. However, the second series *contains* an annuity. Is this statement true or false?
- (2-4) If a firm's earnings per share grew from \$1 to \$2 over a 10-year period, the *total growth* would be 100%, but the *annual growth rate* would be *less than* 10%. True or false? Explain.
- (2-5) Would you rather have a savings account that pays 5% interest compounded semiannually or one that pays 5% interest compounded daily? Explain.

Self-Test Problems Solutions Appear in Appendix A

- (ST-1) Assume that 1 year from now, you will deposit \$1,000 into a savings account that pays 8%.
Future Value
- If the bank compounds interest annually, how much will you have in your account 4 years from now?
 - What would your balance 4 years from now be if the bank used quarterly compounding rather than annual compounding?
 - Suppose you deposited the \$1,000 in 4 payments of \$250 each at Year 1, Year 2, Year 3, and Year 4. How much would you have in your account at Year 4, based on 8% annual compounding?
 - Suppose you deposited 4 equal payments in your account at Year 1, Year 2, Year 3, and Year 4. Assuming an 8% interest rate, how large would each of your payments have to be for you to obtain the same ending balance as you calculated in part a?

- (ST-2) Assume that 4 years from now you will need \$1,000. Your bank compounds interest at an 8% annual rate.
Time Value of Money

- How much must you deposit 1 year from now to have a balance of \$1,000 4 years from now?
- If you want to make equal payments at Years 1 through 4 to accumulate the \$1,000, how large must each of the 4 payments be?
- If your father were to offer either to make the payments calculated in part b (\$221.92) or to give you a lump sum of \$750 1 year from now, which would you choose?
- If you have only \$750 1 year from now, what interest rate, compounded annually, would you have to earn to have the necessary \$1,000 4 years from now?
- Suppose you can deposit only \$186.29 each at Years 1 through 4, but you still need \$1,000 at Year 4. What interest rate, with annual compounding, must you seek out to achieve your goal?
- To help you reach your \$1,000 goal, your father offers to give you \$400 1 year from now. You will get a part-time job and make 6 additional payments of equal amounts each 6 months thereafter. If all of this money is deposited in a bank that pays 8%, compounded semiannually, how large must each of the 6 payments be?
- What is the effective annual rate being paid by the bank in part f?

(ST-3)

Effective Annual Rates

Bank A pays 8% interest, compounded quarterly, on its money market account. The managers of Bank B want its money market account to equal Bank A's effective annual rate, but interest is to be compounded on a monthly basis. What nominal, or quoted, rate must Bank B set?

Easy Problems 1–8

Problems Answers Appear in Appendix B

(2-1)Future Value
of a Single Payment

If you deposit \$10,000 in a bank account that pays 10% interest annually, how much will be in your account after 5 years?

(2-2)Present Value of a
Single Payment

What is the present value of a security that will pay \$5,000 in 20 years if securities of equal risk pay 7% annually?

(2-3)Interest Rate
of a Single Payment

Your parents will retire in 18 years. They currently have \$250,000, and they think they will need \$1,000,000 at retirement. What annual interest rate must they earn to reach their goal, assuming they don't save any additional funds?

(2-4)Number of Periods
of a Single Payment

If you deposit money today in an account that pays 6.5% annual interest, how long will it take to double your money?

(2-5)Number of Periods
for an Annuity

You have \$42,180.53 in a brokerage account, and you plan to deposit an additional \$5,000 at the end of every future year until your account totals \$250,000. You expect to earn 12% annually on the account. How many years will it take to reach your goal?

(2-6)Future Value: Annuity
versus Annuity Due

What is the future value of a 7%, 5-year ordinary annuity that pays \$300 each year? If this were an annuity due, what would its future value be?

(2-7)Present and Future
Value of an Uneven
Cash Flow Stream

An investment will pay \$100 at the end of each of the next 3 years, \$200 at the end of Year 4, \$300 at the end of Year 5, and \$500 at the end of Year 6. If other investments of equal risk earn 8% annually, what is its present value? Its future value?

- (2-8)** Annuity Payment and EAR You want to buy a car, and a local bank will lend you \$20,000. The loan would be fully amortized over 5 years (60 months), and the nominal interest rate would be 12%, with interest paid monthly. What would be the monthly loan payment? What would be the loan's EAR?

Intermediate Problems 9–30

- (2-9)** Present and Future Values of Single Cash Flows for Different Periods Find the following values, *using the equations*, and then work the problems using a financial calculator to check your answers. Disregard rounding differences. (Hint: If you are using a financial calculator, you can enter the known values and then press the appropriate key to find the unknown variable. Then, without clearing the TVM register, you can “override” the variable that changes by simply entering a new value for it and then pressing the key for the unknown variable to obtain the second answer. This procedure can be used in parts b and d, and in many other situations, to see how changes in input variables affect the output variable.)
- An initial \$500 compounded for 1 year at 6%.
 - An initial \$500 compounded for 2 years at 6%.
 - The present value of \$500 due in 1 year at a discount rate of 6%.
 - The present value of \$500 due in 2 years at a discount rate of 6%.

- (2-10)** Present and Future Values of Single Cash Flows for Different Interest Rates Use equations and a financial calculator to find the following values. See the hint for Problem 2-9.
- An initial \$500 compounded for 10 years at 6%.
 - An initial \$500 compounded for 10 years at 12%.
 - The present value of \$500 due in 10 years at a 6% discount rate.
 - The present value of \$500 due in 10 years at a 12% discount rate.

- (2-11)** Time for a Lump Sum to Double To the closest year, how long will it take \$200 to double if it is deposited and earns the following rates? [Notes: (1) See the hint for Problem 2-9. (2) This problem cannot be solved exactly with some financial calculators. For example, if you enter $PV = -200$, $PMT = 0$, $FV = 400$, and $I = 7$ in an HP-12C, and then press the N key, you will get 11 years for part a. The correct answer is 10.2448 years, which rounds to 10, but the calculator rounds up. However, the HP-10B gives the correct answer.]
- 7%.
 - 10%.
 - 18%.
 - 100%.

- (2-12)** Future Value of an Annuity Find the *future value* of the following annuities. The first payment in these annuities is made at the *end* of Year 1; that is, they are *ordinary annuities*. (Notes: See the hint to Problem 2-9. Also, note that you can leave values in the TVM register, switch to “BEG,” press FV, and find the FV of the annuity due.)
- \$400 per year for 10 years at 10%.
 - \$200 per year for 5 years at 5%.
 - \$400 per year for 5 years at 0%.
 - Now rework parts a, b, and c assuming that payments are made at the *beginning* of each year; that is, they are *annuities due*.

- (2-13)** Present Value of an Annuity Find the *present value* of the following *ordinary annuities* (see note to Problem 2-9):
- \$400 per year for 10 years at 10%.
 - \$200 per year for 5 years at 5%.
 - \$400 per year for 5 years at 0%.
 - Now rework parts a, b, and c assuming that payments are made at the *beginning* of each year; that is, they are *annuities due*.

(2-14)
Uneven Cash
Flow Stream

- a. Find the present values of the following cash flow streams. The appropriate interest rate is 8%. (Hint: It is fairly easy to work this problem dealing with the individual cash flows. However, if you have a financial calculator, read the section of the manual that describes how to enter cash flows such as the ones in this problem. This will take a little time, but the investment will pay huge dividends throughout the course. Note, if you do work with the cash flow register, then you must enter $CF_0 = 0$.)

Year	Cash Stream A	Cash Stream B
1	\$100	\$300
2	400	400
3	400	400
4	400	400
5	300	100

- b. What is the value of each cash flow stream at a 0% interest rate?

(2-15)
Effective Rate of Interest

Find the interest rates, or rates of return, on each of the following:

- You *borrow* \$700 and promise to pay back \$749 at the end of 1 year.
- You *lend* \$700 and receive a promise to be paid \$749 at the end of 1 year.
- You borrow \$85,000 and promise to pay back \$201,229 at the end of 10 years.
- You borrow \$9,000 and promise to make payments of \$2,684.80 per year for 5 years.

(2-16)
Future Value for Various
Compounding Periods

Find the amount to which \$500 will grow under each of the following conditions:

- 12% compounded annually for 5 years.
- 12% compounded semiannually for 5 years.
- 12% compounded quarterly for 5 years.
- 12% compounded monthly for 5 years.

(2-17)
Present Value for Various
Compounding Periods

Find the present value of \$500 due in the future under each of the following conditions:

- 12% nominal rate, semiannual compounding, discounted back 5 years.
- 12% nominal rate, quarterly compounding, discounted back 5 years.
- 12% nominal rate, monthly compounding, discounted back 1 year.

(2-18)
Future Value of an
Annuity for Various
Compounding Periods

Find the future values of the following ordinary annuities:

- FV of \$400 each 6 months for 5 years at a nominal rate of 12%, compounded semiannually.
- FV of \$200 each 3 months for 5 years at a nominal rate of 12%, compounded quarterly.
- The annuities described in parts a and b have the same amount of money paid into them during the 5-year period and both earn interest at the same nominal rate, yet the annuity in part b earns \$101.75 more than the one in part a over the 5 years. Why does this occur?

(2-19)
Effective versus Nominal
Interest Rates

Universal Bank pays 7% interest, compounded annually, on time deposits. Regional Bank pays 6% interest, compounded quarterly.

- Based on effective interest rates, in which bank would you prefer to deposit your money?
- Could your choice of banks be influenced by the fact that you might want to withdraw your funds during the year as opposed to at the end of the year? In answering this question, assume that funds must be left on deposit during the entire compounding period in order for you to receive any interest.

- (2-20)** Amortization Schedule
- Set up an amortization schedule for a \$25,000 loan to be repaid in equal installments at the end of each of the next 5 years. The interest rate is 10%.
 - How large must each annual payment be if the loan is for \$50,000? Assume that the interest rate remains at 10% and that the loan is paid off over 5 years.
 - How large must each payment be if the loan is for \$50,000, the interest rate is 10%, and the loan is paid off in equal installments at the end of each of the next 10 years? This loan is for the same amount as the loan in part b, but the payments are spread out over twice as many periods. Why are these payments not half as large as the payments on the loan in part b?
- (2-21)** Growth Rates
- Hanebury Corporation's current sales were \$12 million. Sales were \$6 million 5 years earlier.
- To the nearest percentage point, at what rate have sales been growing?
 - Suppose someone calculated the sales growth for Hanebury Corporation in part a as follows: "Sales doubled in 5 years. This represents a growth of 100% in 5 years, so, dividing 100% by 5, we find the growth rate to be 20% per year." Explain what is wrong with this calculation.
- (2-22)** Expected Rate of Return
- Washington-Pacific invests \$4 million to clear a tract of land and to set out some young pine trees. The trees will mature in 10 years, at which time Washington-Pacific plans to sell the forest at an expected price of \$8 million. What is Washington-Pacific's expected rate of return?
- (2-23)** Effective Rate of Interest
- A mortgage company offers to lend you \$85,000; the loan calls for payments of \$8,273.59 per year for 30 years. What interest rate is the mortgage company charging you?
- (2-24)** Required Lump Sum Payment
- To complete your last year in business school and then go through law school, you will need \$10,000 per year for 4 years, starting next year (that is, you will need to withdraw the first \$10,000 one year from today). Your rich uncle offers to put you through school, and he will deposit in a bank paying 7% interest a sum of money that is sufficient to provide the 4 payments of \$10,000 each. His deposit will be made today.
- How large must the deposit be?
 - How much will be in the account immediately after you make the first withdrawal? After the last withdrawal?
- (2-25)** Repaying a Loan
- While Mary Corens was a student at the University of Tennessee, she borrowed \$12,000 in student loans at an annual interest rate of 9%. If Mary repays \$1,500 per year, how long, to the nearest year, will it take her to repay the loan?
- (2-26)** Reaching a Financial Goal
- You need to accumulate \$10,000. To do so, you plan to make deposits of \$1,250 per year, with the first payment being made a year from today, in a bank account that pays 12% annual interest. Your last deposit will be less than \$1,250 if less is needed to round out to \$10,000. How many years will it take you to reach your \$10,000 goal, and how large will the last deposit be?
- (2-27)** Present Value of a Perpetuity
- What is the present value of a perpetuity of \$100 per year if the appropriate discount rate is 7%? If interest rates in general were to double and the appropriate discount rate rose to 14%, what would happen to the present value of the perpetuity?
- (2-28)** PV and Effective Annual Rate
- Assume that you inherited some money. A friend of yours is working as an unpaid intern at a local brokerage firm, and her boss is selling securities that call for 4 pay-

ments, \$50 at the end of each of the next 3 years, plus a payment of \$1,050 at the end of Year 4. Your friend says she can get you some of these securities at a cost of \$900 each. Your money is now invested in a bank that pays an 8% nominal (quoted) interest rate but with quarterly compounding. You regard the securities as being just as safe, and as liquid, as your bank deposit, so your required effective annual rate of return on the securities is the same as that on your bank deposit. You must calculate the value of the securities to decide whether they are a good investment. What is their present value to you?

(2-29)

Loan Amortization

Assume that your aunt sold her house on December 31 and that she took a mortgage in the amount of \$10,000 as part of the payment. The mortgage has a quoted (or nominal) interest rate of 10%, but it calls for payments every 6 months, beginning on June 30, and the mortgage is to be amortized over 10 years. Now, 1 year later, your aunt must inform the IRS and the person who bought the house of the interest that was included in the two payments made during the year. (This interest will be income to your aunt and a deduction to the buyer of the house.) To the closest dollar, what is the total amount of interest that was paid during the first year?

(2-30)

Loan Amortization

Your company is planning to borrow \$1,000,000 on a 5-year, 15%, annual payment, fully amortized term loan. What fraction of the payment made at the end of the second year will represent repayment of principal?

Challenging Problems 31–34**(2-31)**

Nonannual Compounding

- It is now January 1. You plan to make 5 deposits of \$100 each, one every 6 months, with the first payment being made *today*. If the bank pays a nominal interest rate of 12% but uses semiannual compounding, how much will be in your account after 10 years?
- You must make a payment of \$1,432.02 10 years from today. To prepare for this payment, you will make 5 equal deposits, beginning today and for the next 4 quarters, in a bank that pays a nominal interest rate of 12%, quarterly compounding. How large must each of the 5 payments be?

(2-32)

Nominal Rate of Return

Anne Lockwood, manager of Oaks Mall Jewelry, wants to sell on credit, giving customers 3 months in which to pay. However, Anne will have to borrow from her bank to carry the accounts payable. The bank will charge a nominal 15%, but with monthly compounding. Anne wants to quote a nominal rate to her customers (all of whom are expected to pay on time) that will exactly cover her financing costs. What nominal annual rate should she quote to her credit customers?

(2-33)

Required Annuity Payments

Assume that your father is now 50 years old, that he plans to retire in 10 years, and that he expects to live for 25 years after he retires, that is, until he is 85. He wants his first retirement payment to have the same purchasing power at the time he retires as \$40,000 has today. He wants all his subsequent retirement payments to be equal to his first retirement payment (do not let the retirement payments grow with inflation: he realizes that the real value of his retirement income will decline year by year after he retires). His retirement income will begin the day he retires, 10 years from today, and he will then get 24 additional annual payments. Inflation is expected to be 5% per year from today forward; he currently has \$100,000 saved up; and he expects to earn a return on his savings of 8% per year, annual compounding. To the nearest dollar, how much must he save during each of the next 10 years (with equal deposits being made at the end of each year) to meet his retirement goal? (Hint: Neither the amount he saves nor the amount he withdraws upon retirement is a growing annuity.)

(2-34)
Growing Annuity
Payments

You wish to accumulate \$1 million by your retirement date, which is 25 years from now. You will make 25 deposits in your bank, with the first occurring today. The bank pays 8% interest, compounded annually. You expect to get an annual raise of 3%, so you will let the amount you deposit each year also grow by 3% (i.e., your second deposit will be 3% greater than your first, the third will be 3% greater than the second, etc.). How much must your first deposit be to meet your goal?



Spreadsheet Problem

(2-35)
Build a Model: The Time
Value of Money

Start with the partial model in the file *FM12 Ch 02 P35 Build a Model.xls* from the textbook's Web site. Answer the following questions, using a spreadsheet model to do the calculations.

- Find the FV of \$1,000 invested to earn 10% after 5 years. Answer this question by using a math formula and also by using the *Excel* function wizard.
- Now create a table that shows the FV at 0%, 5%, and 20% for 0, 1, 2, 3, 4, and 5 years. Then create a graph with years on the horizontal axis and FV on the vertical axis to display your results.
- Find the PV of \$1,000 due in 5 years if the discount rate is 10%. Again, work the problem with a formula and also by using the function wizard.
- A security has a cost of \$1,000 and will return \$2,000 after 5 years. What rate of return does the security provide?
- Suppose California's population is 30 million people, and its population is expected to grow by 2% per year. How long would it take for the population to double?
- Find the PV of an annuity that pays \$1,000 at the end of each of the next 5 years if the interest rate is 15%. Then find the FV of that same annuity.
- How would the PV and FV of the annuity change if it were an annuity due rather than an ordinary annuity?
- What would the FV and PV for parts a and c be if the interest rate were 10% with semiannual compounding rather than 10% with annual compounding?
- Find the PV and FV of an investment that makes the following end-of-year payments. The interest rate is 8%.

Year	Payment
1	\$100
2	200
3	400

- Suppose you bought a house and took out a mortgage for \$50,000. The interest rate is 8%, and you must amortize the loan over 10 years with equal end-of-year payments. Set up an amortization schedule that shows the annual payments and the amount of each payment that goes to pay off the principal and the amount that constitutes interest expense to the borrower and interest income to the lender.
 - Create a graph that shows how the payments are divided between interest and principal repayment over time.
 - Suppose the loan called for 10 years of monthly payments, with the same original amount and the same nominal interest rate. What would the amortization schedule show now?



e-resource

Cyberproblem

Please go to the textbook's Web site to access any Cyberproblems.

Mini Case



Assume that you are nearing graduation and that you have applied for a job with a local bank. As part of the bank's evaluation process, you have been asked to take an examination that covers several financial analysis techniques. The first section of the test addresses discounted cash flow analysis. See how you would do by answering the following questions.

- Draw time lines for (1) a \$100 lump sum cash flow at the end of Year 2, (2) an ordinary annuity of \$100 per year for 3 years, and (3) an uneven cash flow stream of $-\$50$, $\$100$, $\$75$, and $\$50$ at the end of Years 0 through 3.
- (1) What is the future value of an initial \$100 after 3 years if it is invested in an account paying 10% annual interest?
(2) What is the present value of \$100 to be received in 3 years if the appropriate interest rate is 10%?
- We sometimes need to find how long it will take a sum of money (or anything else) to grow to some specified amount. For example, if a company's sales are growing at a rate of 20% per year, how long will it take sales to double?
- If you want an investment to double in 3 years, what interest rate must it earn?
- What is the difference between an ordinary annuity and an annuity due? What type of annuity is shown below? How would you change it to the other type of annuity?



- (1) What is the future value of a 3-year ordinary annuity of \$100 if the appropriate interest rate is 10%?
(2) What is the present value of the annuity?
(3) What would the future and present values be if the annuity were an annuity due?
- What is the present value of the following uneven cash flow stream? The appropriate interest rate is 10%, compounded annually.



- (1) Define (a) the stated, or quoted, or nominal rate (I_{NOM}) and (b) the periodic rate (I_{PER}).
(2) Will the future value be larger or smaller if we compound an initial amount more often than annually, for example, every 6 months, or *semi-annually*, holding the stated interest rate constant? Why?

